Fall 2026, Math 296: Week 13 Problem Set
Due: Thursday, December 3rd, 2026
Equivalence Relations and Posets

Discussion problems. The problems below should be worked on in class.

(D1) Compare answers to the preliminary problems, and come to a consensus within your group. You do **not** need to write anything on the board! Did anyone find all 19 partial orders?

(D2) *Hasse diagrams.* Draw a Hasse diagram for each of the following posets P.

(a) The poset $P_1 = \mathbb{N}$ under divisibility, restricted to the set $\{1, 2, \ldots, 15\}$.
Recall: given $a, b \in \mathbb{Z}$, we say $a \mid b$ whenever $b = ka$ for some $k \in \mathbb{Z}$.

(b) The poset $P_2 = \mathcal{P}(\mathbb{N})$ under set containment, restricted to sets containing $B = \{3, 5\}$ and contained in $C = \{2, 3, 4, 5, 6\}$.

(c) The poset $P_3 = \mathbb{N} \times \mathbb{N}$ under the partial order $\preceq$ with $(a, b) \preceq (a', b')$ whenever $a \leq a'$ and $b \leq b'$, restricted to the set $\{(a, b) : 1 \leq a \leq b \leq 5\}$.

(D3) *The full divisibility poset.* Consider the poset on $P = \mathbb{N} \cup \{0\}$ under divisibility.

(a) Draw tha Hasse diagram of P restricted to $\{0, 1, 2, \ldots, 10\}$.

(b) Does P have a minimum? Does P have a maximum?

(c) Locate a chain in P that cannot be extended to a larger chain in P.

(d) Locate an antichain in P that cannot be extended to a larger antichain in P.

(D4) *Equivalence classes.* Fix an equivalence relation $\sim$ on a set A. The *equivalence class* of an element $a \in A$ is the set

$$[a] = \{b \in A : a \sim b\}.$$

(a) For each of the following equivalence relations $\sim$, find all equivalence classes.

(i) The relation $\sim$ on $\mathbb{Z}$ with $a \sim b$ whenever $a = b$.

(ii) The relation $\sim$ on $\mathbb{Z}$ with $a \sim b$ whenever $b - a$ is even.

(iii) The relation $\sim$ on $\mathbb{R}$ with $a \sim b$ whenever $b - a \in \mathbb{Z}$.

(iv) The relation $\sim$ on $\mathcal{P}(\{1, 2, 3\})$ with $B \sim C$ whenever $|B| = |C|$.

(b) Fill in the blanks in the following proof that for any $a, b \in A$, $[a] \cap [b] = \emptyset$ or $[a] = [b]$.

Proof. Suppose $[a] \cap [b] \neq \emptyset$. We must prove _____.
First, we claim $a \sim b$. Since $[a] \cap [b] \neq \emptyset$, there exists $c \in$ _____. This means $a \sim c$ and _____. Since $\sim$ is symmetric, _____ implies $c \sim b$, and since $\sim$ is transitive, _____ and _____ together imply $a \sim b$. This proves the claim.
We now move on to proving $[a] = [b]$. Fix $d \in [b]$. We wish to show _____. Since $d \in [b]$, we have _____. Since $\sim$ is transitive, _____ and _____ imply _____, so _____. Thus, _____.
Having proven $[b] \subseteq [a]$, an analogous argument shows _____, thereby completing the proof that $[a] = [b]$. □

(c) Recall that a *partition* of A is a collection $\{B_i\}$ of nonempty subsets $B_i \subseteq A$ satisfying:

- $B_i \cap B_j = \emptyset$ for all distinct i, j; and
- $A = \bigcup_i B_i$.

Using part (b), prove the set $\{[a] : a \in A\}$ of equivalence classes is a partition of A.

Homework problems. You must submit *all* homework problems in order to receive full credit.

(H1) Fix a function $f : A \to B$. Define a relation $\sim$ on A with $a \sim a'$ whenever $f(a) = f(a')$.

(a) Prove $\sim$ is an equivalence relation.

(b) Fix $a \in A$, and let $b = f(a)$. Identify the equivalence class $[a]$ as a familiar set from a prior week.

(H2) Complete Problems (D4)(b) and (D4)(c).

(H3) Define a relation $\sim$ on $\mathcal{P}(\mathbb{N})$ with $B \sim C$ whenever the symmetric difference $B \Delta C$ is finite.

(a) Prove $\sim$ is an equivalence relation.

(b) Classify which elements of $\mathcal{P}(\mathbb{N})$ lie in $[\emptyset]$. the equivalence class of $\emptyset$.

(c) Classify which elements of $\mathcal{P}(\mathbb{N})$ lie in $[\mathbb{N}]$, the equivalence class of $\mathbb{N}$.

(d) Locate an element $B \in \mathcal{P}(\mathbb{N})$ with $[B] \neq [\emptyset]$ and $[B] \neq [\mathbb{N}]$.