A finiteness result for local cohomology of Stanley-Reisner rings

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Local Cohomology

R is a commutative Noetherian ring with 1.

Let $\mathbf{f} = f_1, \ldots, f_l$ be generators for an ideal $I \subset R$.

The Čech complex on $\mathbf{f}$ is the complex

$$\check{C}(\mathbf{f}; R) : 0 \to R \to \bigoplus_{i=1}^{l} R_{f_i} \to \bigoplus_{i<j} R_{f_i f_j} \to \cdots \to R_{f_1 \cdots f_l} \to 0.$$

The i-th local cohomology of M with support in I is

$$H_I^i(M) = H^i(\check{C}^\bullet(\mathbf{f}; R) \otimes_R M) = \check{H}^i(\mathbf{f}; M).$$

Background

- The Bass numbers over a Stanley-Reisner ring of $H^i_I(M)$ are finite. (Helm and Miller)
- The local cohomology of the following rings have finitely many associated primes:
  - local regular rings of characteristic 0 (Lyubeznik)
  - regular rings of prime characteristic (Huneke, Sharp)
  - smooth $\mathbb{Z}$-algebras
    (Bhatt, Blickle, Lyubeznik, Singh, Zhang.)

Stanley-Reisner Rings

Let $\mathbb{k}$ be a field.

$S = \mathbb{k}[x_1, \ldots, x_n]$

Δ is a simplicial complex on the vertex set $V = \{x_1, \ldots, x_n\}$.

The Stanley-Reiser ideal of Δ in S is

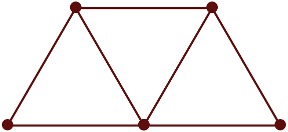
$$I_\Delta = \langle x_{i_1} \cdots x_{i_r} : \{x_{i_1}, \ldots, x_{i_r}\} \notin \Delta\} \rangle.$$

The Stanley-Reisner ring of Δ in S is

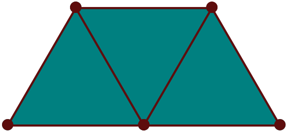
$$\mathbb{k}[\Delta] = S / I_\Delta.$$

Simplicial Complexes

We call Δ a T-space if for each face $F \in \Delta$ and each vertex $v \notin F$, there is a facet H containing F and not containing $\{v\}$. We say F may be separated from $\{v\}$.



T-space



not a T-space

The star of a face $F \in \Delta$ is $\mathrm{star}_\Delta(F) = \{G \in \Delta : F \cup G \in \Delta\}$. The core of V is $\mathrm{core}(V) = \{v \in V : \mathrm{star}_\Delta\, v \neq V\}$. The core of Δ is $\mathrm{core}(\Delta) = \Delta_{\mathrm{core}\, V}$.

Rings of differential operators

Differential operators on R are defined inductively as follows:

- for each $r \in R$, the multiplication by r map $r : R \to R$ is a differential operator of order 0.
- for $n > 0$, the differential operators or order less than or equal to n are the additive maps $\delta : R \to R$ whose commutator $[r, \delta] = r \circ \delta - \delta \circ r$ is a differential operator of order less than or equal to $n - 1$.

Denote the ring of differential operators on R by $\mathcal{D}(R)$.

If R is an A-algebra, $\mathcal{D}(R; A)$ is the ring of A-linear differential operators.

Rings of differential operators

Set $\partial_i = \frac{\partial}{\partial x_i}$ to be the derivative with respect to x_i.

Theorem (-, Madsen, Wheeler)

If $\mathrm{core}(\Delta)$ is a T-space, then the ring of differential operators on $\mathbb{k}[\Delta]$ is

$$\mathcal{D} = \mathcal{D}(\mathbb{k}[\Delta]; \mathbb{k}) = \mathbb{k}[\Delta]\langle x_i \partial_i^t : 1 \leq i \leq n, 0 \leq t \rangle.$$

The ring of differential operators on $\mathbb{k}[x_1, \ldots, x_n]$ is the n-th Weyl algebra over $\mathbb{k}$:

$$\mathcal{D}_n = \mathbb{k}\langle x_1 \ldots, x_n, \partial_1, \ldots, \partial_n : [x_i, x_j] = [\partial_i, \partial_j] = 0, [\partial_i, x_j] = \delta_{ij} \rangle.$$

$\mathcal{D}$-modules

The Bernstein filtration on $\mathcal{D}_n$ is the filtration

$$\mathcal{F} = F_0 \subset F_1 \subset F_2 \subset \cdots$$

where $F_i = \mathbb{k} \cdot \{x^{\mathbf{a}}\partial^{\mathbf{b}} : \sum_j a_j + \sum_k b_k \leq i\}$.

A $\mathbb{k}$-filtration on a $\mathcal{D}$-module M is an ascending chain of $\mathbb{k}$-vector spaces

$$M_0 \subset M_1 \subset M_2 \subset \cdots$$

satisfying

1. $\bigcup_i M_i = M$,
2. for all i and j, $F_i M_j \subset M_{i+j}$.

$\mathcal{D}$-modules

A $\mathcal{D}$-module M is holonomic if it has a $\Bbbk$-filtration $\mathcal{M} = M_0 \subset M_1 \subset \cdots$ such that for all i, $\dim(M_i) \leq Ci^n$.

Lemma (-, Madsen, Wheeler)

Let $\mathrm{core}(\Delta)$ be a T-space and let $\Bbbk[\Delta]$. Then $\Bbbk[\Delta]$ is a holonomic $\mathcal{D}$-module.

Theorem (-, Madsen, Wheeler)

Let $\mathrm{core}(\Delta)$ be a T-space. Then for any $f \in \Bbbk[\Delta]$, $\Bbbk[\Delta]_f$ is a holonomic $\mathcal{D}$-module.

Proof outline of Theorem:

- Let $d = \deg(f)$.
- For each $k \geq 0$, let $M_k = \bigoplus_{i=0}^{k} R_i$ where R_i is the set of homogeneous elements of degree i in R under the standard grading. Then $\mathcal{M} = M_0 \subset M_1 \subset \cdots$ is a filtration.
- Define $M'_k = \mathbb{k}\{\frac{g}{f^k} : g \in M_{k(d+1)}\}$. Then $\mathcal{M}' = M'_0 \subset M'_1 \subset \cdots$ is a $\mathbb{k}$-filtration.
- Since $\mathcal{M}'$ is a $\mathbb{k}$-filtration, then

$$\dim_{\mathbb{k}}(M'_k) \leq \dim_{\mathbb{k}}(M_k(d+1)) \leq C(k(d+1))^r.$$

- Set $C' = C(d+1)^r$.

Finiteness result

Theorem (-, Madsen, Wheeler)

Let $\mathrm{core}(\Delta)$ be a T-space and let I be an ideal in $\Bbbk[\Delta]$. Then $H^i_I(\Bbbk[\Delta])$ has finitely many associated prime ideals.

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THANK YOU!