EKT of Some Wonderful Compactifications

and recent results on Complete Quadrics.
(Based on joint works with Soumya Banerjee and Michael Joyce)

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What is a spherical variety?

Let

- G be a reductive group, $B \subseteq G$ be a Borel subgroup,
- X be a G-variety,
- and $\mathcal{B}(X)$ denote the set of B-orbits in X.

If $|\mathcal{B}(X)| < \infty$, then X is called a spherical G-variety.

Examples.

Simple examples include

- toric varieties,
- (partial) flag varieties,
- symmetric varieties,
- linear algebraic monoids.

There are many other important examples..

Our goal is to present a description of the equivariant K-theory for all smooth projective spherical varieties and record some recent progress on the geometry of the variety of complete quadrics.

Equivariant Chow groups

Let X be a projective nonsingular spherical G-variety and $T \subseteq G$ be a maximal torus.

Theorem (Brion '97)

The map $i^* : A_T^*(X)_{\mathbb{Q}} \to A_T^*(X^T)_{\mathbb{Q}}$ is injective. Moreover, the image of i^* consists of families $(f_x)_{x \in X^T}$ such that

- $f_x \equiv f_y \mod \chi$ *whenever x and y are connected by a T-curve with weight χ.*
- $f_x - 2f_y + f_z \equiv 0 \mod \alpha^2$ *whenever α is a positive root, x, y, z lie in a connected component of $X^{\ker(\alpha)}$ which is isomorphic to $\mathbb{P}^2$.*
- $f_x - f_y + f_z - f_w \equiv 0 \mod \alpha^2$ *whenever α is a positive root, x, y, z, w lie in a connected component of $X^{\ker(\alpha)}$ which is isomorphic to a rational ruled surface.*

Remarks

The underlying idea in Brion's result is to study the fixed loci of all codimension one subtori $S \subset T$. This point is exploited by Vezzosi and Vistoli for K-theory:

Theorem (Vezzosi-Vistoli '03)

Suppose D is a diagonalizable group acting on a smooth proper scheme X defined over a perfect field; denote by T the toral component of D, that is the maximal subtorus contained in D. Then the restriction homomorphism on K-groups $K_{D,*}(X) \to K_{D,*}(X^T)$ is injective, and its image equals the intersection of all images of the restriction homomorphisms $K_{D,*}(X^S) \to K_{D,*}(X^T)$ for all subtori $S \subset T$ of codimension 1.

Therefore, for a spherical G-variety X, we need to analyze X^S. when S is a codimension one subtorus of T.

Equivariant K-theory

Let k denote the underlying base field that our schemes are defined over and let $R(T)$ denote the representation ring of T.

Theorem (Banerjee-Can, around 2013, posted in 2016)

The T-equivariant K-theory $K_{T,*}(X)$ is isomorphic to the ring of ordered tuples

$$(f_x)_{x \in X^T} \in \prod_{x \in X^T} K_*(k) \otimes R(T)$$

satisfying the following congruence relations:

Equivariant K-theory

Theorem (Banerjee-Can '13, continued)

1) If there exists a T-stable curve with weight χ connecting $x, y \in X^T$, then

$$f_x - f_y = 0 \mod (1 - \chi).$$

Equivariant K-theory

Theorem (Banerjee-Can '13, continued)

2) If there exists a root χ such that an irreducible component $Y \subseteq X^{\ker \chi}$ isomorphic to $Y \simeq \mathbb{P}^2$ connects $x, y, z \in X^T$, then

$$\begin{aligned} f_x - f_y &= 0 \mod (1-\chi), \\ f_x - f_z &= 0 \mod (1-\chi), \\ f_y - f_z &= 0 \mod (1-\chi^2). \end{aligned}$$

Moreover, in this case, there is an element in the Weyl group of (G, T) that fixes x and permutes y and z.

Equivariant K-theory

Theorem (Banerjee-Can '13, continued)

3) If there exists a root χ such that an irreducible component $Y \subseteq X^{\ker \chi}$ isomorphic to $Y \simeq \mathbb{P}^1 \times \mathbb{P}^1$ connects $x, y, z, t \in X^T$, then

$$
\begin{aligned}
f_x - f_y &= 0 \mod (1-\chi), \\
f_y - f_z &= 0 \mod (1-\chi), \\
f_z - f_t &= 0 \mod (1-\chi), \\
f_x - f_t &= 0 \mod (1-\chi).
\end{aligned}
$$

Moreover, in this case, there is an element in the Weyl group of (G, T) that fixes two and permutes the other two.

Equivariant K-theory

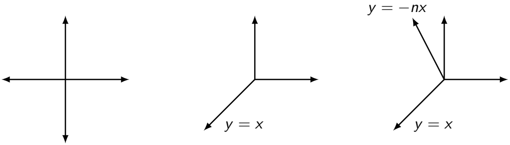
Theorem (Banerjee-Can '13, continued)

4) If there exists a root χ such that an irreducible component $Y \subseteq X^{\ker \chi}$ isomorphic to a Hirzebruch surface F_n that connects $x, y, z, t \in X^T$, then

$$\begin{aligned} f_x - f_y &= 0 \mod (1-\chi), \\ f_z - f_t &= 0 \mod (1-\chi), \\ f_y - f_z &= 0 \mod (1-\chi^{2n}), \\ f_x - f_t &= 0 \mod (1-\chi^{n}). \end{aligned}$$

Moreover, in this case, there is an element in the Weyl group of (G, T) that fixes the points x and t and permutes z and y.

Equivariant K-theory



Δ_1 for $\mathbb{P}_1 \times \mathbb{P}_1$ Δ_1 for $\mathbb{P}(\mathfrak{sl}_2)$ Δ_1 for F_n, $n \geq 1$

Figure: Fans of the irreducible components $Y \subset X^S$

Equivariant K-theory

Theorem (Banerjee-Can '13, continued)

Since $W = W(G,T)$ acts on X^T, it induces an action on $\prod_{x \in X^T} K_*(k) \otimes R(T)$.

The G-equivariant K-theory of X is given by the space of invariants:

$$K_{G,*}(X) = K_{T,*}(X) \cap \left(\prod_{x \in X^T} K_{T,*}(k) \otimes R(T) \right)^W .$$

Applications to wonderful compactifications

- k: algebraically closed, characteristic 0;
- G: semisimple simply-connected algebraic group;
- $\theta : G \to G$ an involutory automorphism;
- $H = G^\theta$: the fixed point subgroup;
- $\tilde{H}$: the normalizer of H in G.

Prolongement magnifique de Demazure

- $\mathfrak{g} = \mathrm{Lie}(G)$,
- $\mathfrak{h} = \mathrm{Lie}(H)$ with $d = \dim \mathfrak{h}$,
- $Gr(\mathfrak{g}, d)$: grassmannian of d dimensional vector subspaces of $\mathfrak{g}$,
- $[\mathfrak{h}]$: the point corresponding to $\mathfrak{h} \subset \mathfrak{g}$,

The wonderful compactification $X_{G/H}$ of $G/\tilde{H}$ is the Zariski closure of the orbit

$$\overline{G \cdot [\mathfrak{h}]} \subset Gr(\mathfrak{g}, d).$$

General properties of $X_{G/H}$

De Concini-Procesi '82:

- $X_{G/H}$ is smooth, complete, and G-spherical.
- The open orbit is $G/H \hookrightarrow X_{G/H}$.
- There are finitely many boundary divisors $X^{\{\alpha\}}$ which are G-stable and indexed by elements of a system of simple roots, $\alpha \in \Delta_{G/H}$.
- Each G-orbit closure is of the form $X^I := \pitchfork_{\alpha \in I} X^{\{\alpha\}}$ for a subset $I \subseteq \Delta_{G/H}$ and moreover

$$X^I \subseteq X^J \iff J \subseteq I.$$

- There exists a unique closed G-orbit $X^{\Delta_{G/H}}$ which is necessarily of the form G/P for some parabolic subgroup P.

First example, the group case.

Let $\mathbf{G} = G \times G$ and $\theta : \mathbf{G} \to \mathbf{G}$ be the automorphism

$$\theta(g_1, g_2) = (g_2, g_1).$$

The fixed subgroup $\mathbf{H}$ is the diagonal copy of G in $\mathbf{G}$.

- The open orbit is $\mathbf{G}/\mathbf{H} \cong G$.
- The closed orbit is isomorphic to $G/B \times G/B^-$, where B^- is the opposite Borel.

Back to the general case.

Let G/P denote the closed orbit in $X_{G/H}$.

- P: parabolic subgroup opposite to $\theta(P)$
- $L = P \cap \theta(P)$
- $T \subset L$ a maximal torus
- $T_0 = \{t \in T : \theta(t) = t\}$
- $T_1 = \{t \in T : \theta(t) = t^{-1}\}$
- W_G, W_H, W_L associated Weyl groups
- Φ_G, Φ_H, Φ_L the root systems of $(G, T), (H, T_0), (L, T)$
- The rank of G/H is $rank(G/H) := \dim T_1$

Spherical pairs of minimal rank

G/H is called of minimal rank if

$$rank(G/H) + rank(H) = rank(G).$$

Geometry of these varieties are studied by Tchoudjem in '05 and Brion-Joshua in '08.

Theorem (Ressayre '04)

Irreducible minimal rank spherical pairs (G, H) with G semisimple and H simple are

- $(\mathbf{G}, \mathbf{H})$ *with* $\mathbf{H}$ *simple.*
- (SL_{2n}, Sp_n).
- (SO_{2n}, SO_{2n-1}).
- (E_6, F_4).
- (SO_7, G_2).

Little Weyl group

Let $\mathcal{X}(T)$ denote the character group of T.

- If $p : \mathcal{X}(T) \to \mathcal{X}(T_1^0)$ is the restriction map, then

$$\Phi_{G/H} := p(\Phi_G) - \{0\}$$

  is a root system, which is possibly non reduced.
- $\Delta_{G/H} = \{\alpha - \theta(\alpha) : \ \alpha \in \Delta_G - \Delta_L\}$ is a basis for $\Phi_{G/H}$.
- The little Weyl group of G/H is defined as

$$W_{G/H} := N_G(T_1^0)/Z_G(T_1^0).$$

Wonderful toric variety

There is a natural torus embedding $T/T_0 \hookrightarrow G/H$. The closure $Y := \overline{T/T_0} \subset X_{G/H}$ is a smooth projective toric variety. Furthermore, $Y = W_{G/H} \cdot Y_0$, where Y_0 is the affine toric subvariety of Y associated with the positive Weyl chamber dual of $\Delta_{G/H}$. Y_0 has a unique T-fixed point, denoted by z_0.

Lemma (Brion-Joshua, Tchoudjem)

(i) *The T-fixed points of $X_{G/H}$ (resp. of Y) are exactly the points $w \cdot z_0$ where $w \in W_G/W_L$ (resp. $w \in W_H/W_L = W_{G/H}$).*

(ii) *For any positive root $\alpha \in \Phi_G^+ \setminus \Phi_L^+$, there exists unique irreducible T-stable curve $C_{\alpha \cdot z_0}$ connecting z_0 and α_{z_0}. The torus T acts on $C_{\alpha \cdot z_0}$ via the character α. This curve is isomorphic to $\mathbb{P}^1$ and we call it as a Type 1 curve.*

(iii) *For any simple root $\gamma = \alpha - \theta(\alpha) \in \Delta_{G/H}$, there exists unique irreducible T-stable curve $C_{\gamma \cdot z_0}$ connecting z_0 and $s_\alpha s_{\theta(\alpha)} \cdot z_0$. The torus T acts on $C_{\gamma \cdot z_0}$ by the character γ. This curve is isomorphic to $\mathbb{P}^1$ and we call it by a Type 2 curve.*

(iv) *The irreducible T-stable curves in X are precisely the W_G-translates of the curves $C_{\alpha \cdot z_0}$ and $C_{\gamma \cdot z_0}$. They are all isomorphic to $\mathbb{P}^1$.*

(v) *The irreducible T-stable curves in Y are the $W_{G/H}$-translates of the curves $C_{\gamma \cdot z_0}$.*

Corollary

The minimal rank wonderful compactification $X_{G/H}$ is T-skeletal; there are finitely many T-fixed points and finitely many T-stable curves.

Using these observations, in 2008 Brion and Joshua obtained a concrete description of the equivariant Chow ring of a wonderful compactification of minimal rank.

We do the same (actually to a finer degree) with the equivariant algebraic K-theory.

Corollary (Banerjee-Can '13)

The T-equivariant K-theory $K_{T,*}(X_{G/H})$ of $X_{G/H}$ is isomorphic to the space of tuples $(f_{w\cdot z_0}) \in \prod_{w\in W_G/W_L} K_*(k)\otimes R(T)$ such that

$$f_{w\cdot z_0} - f_{w'\cdot z_0} = \begin{cases} 0 \mod (1-\alpha) & \textit{if } w^{-1}w' = s_\alpha \\ 0 \mod (1-\alpha\cdot\theta(\alpha)^{-1}) & \textit{if } w^{-1}w' = (s_\alpha \cdot s_{\theta(\alpha)})^{\pm} \end{cases}.$$

The T-equivariant K-theory $K_{T,*}(Y)$ of the toric variety Y is isomorphic to the space of tuples $(f_{w\cdot z_0}) \in \prod_{w\in W_H/W_L} K_*(k)\otimes R(T)$ such that

$$f_{w\cdot z_0} - f_{w'\cdot z_0} = 0 \mod (1-\alpha\cdot\theta(\alpha)^{-1}) \textit{ and } w^{-1}w' = (s_\alpha \cdot s_{\theta(\alpha)})^{\pm}.$$

Let W^H denote the minimal coset representatives of W_G/W_H.

Theorem (Banerjee-Can '15)

There is an isomorphism of rings

$$\prod_{w \in W^H} K_{T,*}(Y) \cong K_{T,*}(X_{G/H}).$$

Moreover this is an isomorphism of $K_*(k) \otimes R(T)$ modules.

Corollary (Banerjee-Can '15)

The G-equivariant K-theory $K_{G,*}(X_{G/H})$ of $X_{G/H}$ is isomorphic to W_H-invariants of the T-equivariant K-theory of the toric variety Y.

A non T-skeletal example

Consider the automorphism $\theta : SL_n \to SL_n$ defined by $\theta(g) = (g^{-1})^\top$. Then the fixed subgroup of θ is SO_n, hence the symmetric variety G/G^θ is

$$G/H := SL_n/SO_n.$$

The maximal torus of diagonal matrices in SL_n is unisotropic with respect to θ; $T = T_1$, hence the set of restricted simple roots $\Delta_{G/H}$ is the root system of (SL_n, T).

Let X_0 denote the open set of the projectivization of Sym_n, the space of symmetric n-by-n matrices, consisting of matrices with non-zero determinant. Elements of X_0 should be interpreted as (the defining equations of) smooth quadric hypersurfaces in $\mathbb{P}^{n-1}$. The group SL_n acts on X_0 by change of variables defining the quadric hypersurfaces, which translates to the action

$$g \cdot A = gAg^\top$$

on Sym_n.

X_0 is a homogeneous space under this SL_n action and the stabilizer of the quadric $x_1^2 + x_2^2 + \cdots + x_n^2 = 0$ (equivalently, the class of the identity matrix) is the normalizer group of SO_n in SL_n, which we denote by $\widetilde{SO}_n$.

Definition

The variety of complete quadrics X_n is the wonderful compactification of $SL_n/\widetilde{SO}_n$.

Its classical definition (Schubert 1879) is as follows. A point $\mathcal{P} \in X_n$ is described by the data of a flag

$$\mathcal{F} : V_0 = 0 \subset V_1 \subset \cdots \subset V_{s-1} \subset V_s = \mathbb{C}^n \tag{1}$$

and a collection $\mathcal{Q} = (Q_1, \dots Q_s)$ of quadrics, where Q_i is a quadric in $\mathbb{P}(V_i)$ whose singular locus is $\mathbb{P}(V_{i-1})$.

There are alternative descriptions of X_n:

Theorem (Semple 1948)

X_n is the closure of the image of the map

$$[A] \mapsto ([A], [\Lambda^2(A)], \ldots, [\Lambda^{n-1}(A)]) \in \prod_{i=1}^{n-1} \mathbb{P}(\Lambda^i(Sym_n)).$$

Theorem (Vainsencher 1982)

X_n can be obtained by the following sequence of blow-ups: in the naive compactification $\mathbb{P}^{n-1}$ of X_0, first blow up the locus of rank 1 quadrics; then blow up the strict transform of the rank 2 quadrics; …; then blow up the strict transform of the rank $n-1$ quadrics.

The closed SL_n-orbit in X_n is SL_n/B and the dense open orbit is $SL_n/\widetilde{SO}_n$. To describe the geometry of X_n, first, one needs to understand the combinatorics of Borel orbits in X_n. Notation:

- A composition of n is an ordered sequence $\mu = (\mu_1, \dots, \mu_k)$ of positive integers that sum to n.
- Define set(μ) of a composition by

$$\mu = (\mu_1, \dots, \mu_k) \leftrightarrow \mathrm{set}(\mu) := \{\mu_1, \mu_1 + \mu_2, \dots, \mu_1 + \cdots + \mu_{k-1}\},$$

  This yields an equivalent parameterization of the G-orbits of X_n.
- The G-orbit associated with the composition μ is denoted by O^μ.

Let μ' and μ be two compositions of n. In Zariski topology

$$O^{\mu'} \subseteq \overline{O^{\mu}} \iff \mathrm{set}(\mu) \subseteq \mathrm{set}(\mu').$$

The B-orbits of X_n lying in the open orbit $O^{(n)}$ are parametrized by I_n, the set of involutions in S_n.

More generally, (as noticed by Springer '04) the B-orbits in O^{μ} are parameterized by combinatorial objects that we call μ-*involutions*. Concisely, a μ-involution is a permutation of the set $[n]$ written in one-line notation and partitioned into strings by μ, so that each string is an involution with respect to the relative ordering of its numbers. For example, $[26|8351|7|94]$ is a $(2,4,1,2)$-involution and the string 8351 is equivalent to the involution 4231.

We denote by I_μ the set of μ-involutions. The identity μ-involution, whose entries are given in the increasing order, is the representative of the dense B-orbit in the G-orbit O^μ. At the other extreme, the B-orbits in the closed orbit are parametrized by permutations and the inclusion relations among B-orbit closures is just the opposite of the well-known Bruhat-Chevalley ordering (so that the identity permutation corresponds to the dense B-orbit).

Associated to a μ-involution π is a distinguished complete quadric Q_π. Viewed as a permutation, $\pi \in \mathrm{I}_\mu$ has the decomposition $\pi = uv$ with $u \in S_\mu$ and $v \in S^\mu$, where S^μ is the minimal length right coset representatives of the parabolic subgroup S_μ in S_n.

Suppose $\mu = (\mu_1, \ldots, \mu_k)$ and let e_i denote the i-th standard basis vector of $\mathbb{C}^n$. Then the desired flag of Q_π is given by the subspaces V_i, $i = 1, 2, \ldots, k$, which are spanned by $e_{\pi(j)}$ for $1 \leq j \leq \mu_1 + \mu_2 + \cdots + \mu_i$.

To construct the corresponding sequence of smooth quadrics, consider $(u_1, u_2, \ldots, u_k)$, the image of u under the isomorphism $S_\mu \cong S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_k}$. Since π is a μ-involution, each $u_i \in \mathrm{I}_{\mu_i}$. Then the smooth quadric in $\mathbb{P}(V_i/V_{i-1})$ that defines Q_π is given by the symmetric matrix in the permutation matrix representation of u_i.

Lemma (Banerjee-Can-Joyce '16)

Let $\pi = [\pi_1|\cdots|\pi_k]$ be a μ-involution and let Y_π be the corresponding B-orbit. Then

Y_π has a T-fixed point if and only if for $i = 1, \ldots, k$ the length of π_i (as a string) is at most 2; if $\pi_i = i_1 i_2$ for numbers $i_1, i_2 \in [n]$, then $i_1 > i_2$ (hence π_i corresponds to the nonsingular quadric $x_{i_1} x_{i_2}$).

Definition

We call a μ-involution as in the above lemma a *barred permutation*. The number of barred permutations of $[n]$ is denoted by t_n, $n \geq 1$. By convention we set $t_0 = 1$. The set of all barred permutations on $[n]$ is denoted by $B(S_n)$.

Theorem (Banerjee-Can-Joyce '16)

The exponential generating series $F_{exp}(x) := \sum_{n\geq 0} \frac{t_n}{n!} x^n$ *of the number of* T*-fixed points in* X_n *is given by*

$$F_{exp}(x) = \frac{1 + x - x^2/2 - x^3/2}{(1 - x - x^2/2)^2} = \frac{-(x+1)(x^2-2)}{(2x^2+4x-4)^2}.$$

Corollary (Banerjee-Can-Joyce '16)

The number of T*-fixed points in* X_n *is equal to*

$$a_n(n+1)! + a_{n-1}n!,$$

where

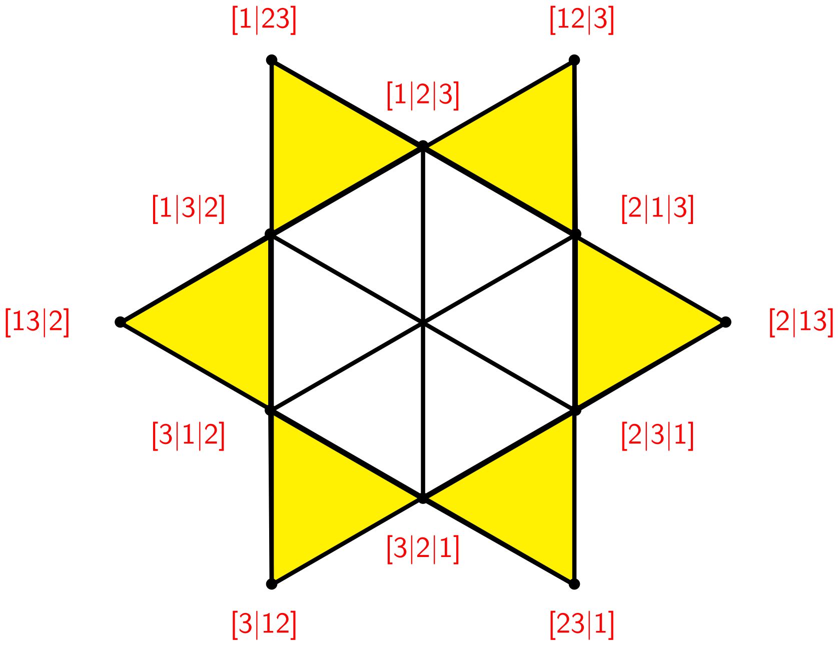
$$a_n = \begin{cases} \frac{\sum_{i=0}^{n/2} \binom{n+1}{2i+1} 3^i}{2^n} & \text{if } n+1 = 2m+1; \\ \frac{\sum_{i=0}^{(n-1)/2} \binom{n+1}{2i+1} 3^i}{2^n} & \text{if } n+1 = 2m. \end{cases}$$

Our next task is to understand the T-stable surfaces and curves in X_n:

Theorem (Banerjee-Can-Joyce '16)

An irreducible component of X^S, the fixed locus of a codimension-one subtorus of T is either a $\mathbb{P}^1$ or a $\mathbb{P}^2$.

We can tell exactly how do these $\mathbb{P}^1$'s and $\mathbb{P}^2$'s fit together. For example, when $n = 3$ we have: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Theorem (Banerjee-Can-Joyce '16)

Let $T \subset G = SL_n$ denote maximal torus of diagonal matrices. The T equivariant K-theory $K_{T,*}(X_n)$ is isomorphic to the ring of tuples $(f_x) \in \prod_{x \in B(S_n)} K_*(k) \otimes R(T)$ satisfying the following congruence conditions:

- $f_x - f_y = 0 \mod (1-\chi)$ *when x, y are connected by a T stable curve with weight χ.*
- $f_x - f_y = f_x - f_z = 0 \mod (1-\chi)$ *and* $f_y - f_z = 0 \mod (1-\chi^2)$, *$\chi$ is a root and x, y, z lie on a component of the subvariety $X_n^{\ker(\chi)}$, which is isomorphic to $\mathbb{P}^2$. There is a permutation that fixes x and permutes y and z.*

Moreover, the symmetric group S_n acts on the torus fixed point set X_n^T by permuting them and the G equivariant K-theory is given by the space of S_n-invariants in $K_{T,*}(X_n)$.

Define $\tau : \{\mu\text{-involutions}\} \to \{\text{barred permutations}\}$ as follows: Suppose $\pi = \pi_1|\pi_2|\dots|\pi_k$. For each π_j, order its cycles in lexicographic order on the largest value in each cycle. Then add bars between each cycle. Since π is a μ-involution, every cycle that occurs in each π_j has length one or two. Finally, convert one-cycles (i) into the numeral i and two-cycles (ij) with $i < j$ into the string ji. For example,

$$\tau((68)|(25)(4)(9)|(13)(7)) = [86|4|52|9|31|7].$$

Theorem (Banerjee-Can-Joyce '16)

There is a 1-PSG λ such that for any μ-involution π, the limit $\lim_{t\to 0} \lambda(t) \cdot Q_\pi$ is the T-fixed quadric parameterized by $\tau(\pi)$.

We now wish to define a map

$$\sigma : \{\text{barred permutations}\} \to \{\mu\text{-involutions}\}$$

which will have the following geometric interpretation.

Let Q_α be the T-fixed quadric associated to a barred permutation α. Then $\sigma(\alpha)$ will correspond to the distinguished quadric in the dense B-orbit of the cell that contains Q_α. In other words, the B-orbit of $Q_{\sigma(\alpha)}$ will have the largest dimension among all B-orbits that flow to Q_α.

First, we define the notion of ascents and descents in a barred permutation $\alpha = [\alpha_1|\alpha_2|\dots|\alpha_k]$. First, define d_j to be the largest value occurring in α_j, giving rise to a sequence $\mathbf{d} = (d_1, d_2, \dots, d_k)$. For example, if $\alpha = [86|9|52|4|7|31]$, then $\mathbf{d} = (8, 9, 5, 4, 7, 3)$. We say that π has a descent (resp., ascent) at position i if $\mathbf{d}$ has a descent (resp., ascent) at position i.

The μ-involution $\sigma(\alpha)$ is constructed by first converting strings i of length 1 into one-cycles (i) and strings ji of length 2 into two-cycles (ij). Then remove the bars at positions of ascent and keep the bars at positions of descent in α. For example,

$$\sigma([86|4|52|9|31|7]) = (68)|(25)(4)(9)|(13)(7).$$

Theorem (Banerjee-Can-Joyce '16)

For any barred permutation α, the B-orbit of $Q_{\sigma(\alpha)}$ has the largest dimension among all B-orbits that flow to Q_α.

We illustrate the resulting cell decomposition when $n = 3$ in the next figure. The dimension of a cell corresponding to a vertex in the figure is equal to the length of any chain from the bottom cell. A vertex corresponding to cell C is connected by an edge to a vertex of a cell C' of one dimension lower if and only if C' is contained in the closure of C.

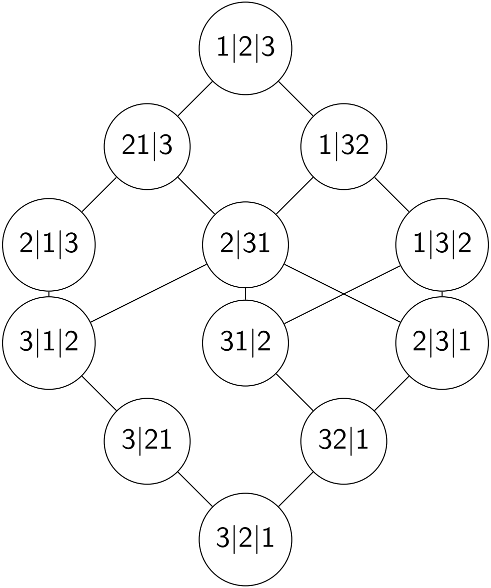
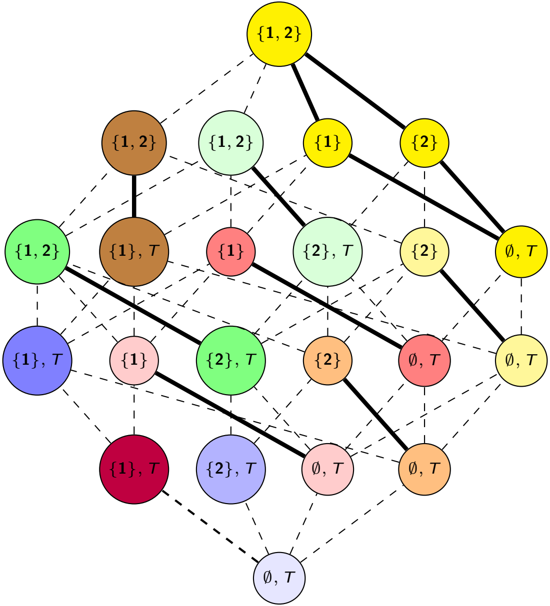


Figure: Cell decomposition of the complete quadrics for $n = 3$. The labels give the barred permutation parametrizing the T-fixed point in the cell.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Given a barred permutation α, let $w(\alpha)$ denote the permutation in one-line notation that is obtained by removing all bars in α. Let $\mathrm{inv}(\alpha)$ denote the number of length 2 strings that occur and let $\mathrm{asc}(\alpha)$ denote the number of ascents in α.

Theorem (Banerjee-Can-Joyce '16)

The dimension of the cell containing the T-fixed quadric parameterized by α is $\ell(w_0) - \ell(w(\alpha)) + inv(\alpha) + asc(\alpha)$.

We found an algorithm to decide when two cells closures are contained in each other by describing the Bruhat-Chevalley ordering on the Borel orbits contained in the same G-orbit + by using W-sets of Brion.

Appendix

Definition of algebraic K-theory:

- $\mathcal{C}$: a small category;
- $B\mathcal{C}$: the classifying complex of $\mathcal{C}$, which, by definition, is the topological realization of the simplicial complex whose simplicies are chains of morphisms.

Definition

• nth K-group of $\mathcal{C}$ is the nth homotopy group of $B\mathcal{C}$.
• If X is a G-variety, then its nth G-equivariant K-group is the nth K-group of the (small) category of G-equivariant vector bundles on X.