Cohen-Kaplansky Domains

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April 17, 2016

Introduction to CK domains I

In 1946, Cohen and Kaplansky introduced the class of domains now known as Cohen-Kaplansky domains. A *Cohen-Kaplansky domain* is an atomic domain that contains only finitely many irreducibles up to associates. As an extension of the above definition, we define a CK-n domain to be a CK-domain containing precisely n irreducible elements and we define a CK*-n domain to be a CK-n domain where every irreducible element is a nonprime. Implicit in their seminal work was the question as to the existence of CK*-n domains for every positive integer n. This question was mentioned explicitly by D. D. Anderson in a later paper. Notice it is easy to construct a CK-n domain for every n simply by localizing $\mathbb{Z}$ at a set of n distinct primes.
Later D. D. Anderson and J. L. Mott looked at the infinite collection of D+M constructions of the form $R = k + x^m K[[x]]$

Introduction to CK domains II

with $k \subseteq K$ finite fields, $m \in \mathbb{N}$. For these constructions, the closed form formula for determining the number of irreducibles is given by $n = m|\frac{K^*}{k^*}||K|^{m-1}$. This produces many examples of CK*-n domains for small values of n. For instance, for values of $n < 250$, this construction produces CK*-n domains for over 100 distinct values of n. The smallest value of n that cannot be constructed is $n = 11$. The exponential nature of this formula makes it clear that the density of the output decreases with larger values of n.

A similar line of thinking produces the following kind of construction. Let $a_1 < a_2 < \cdots < a_k \in \mathbb{N}$, $q = p^n$ a prime power, and $\mathbb{F}_q$ the field of q elements. The construction

$$\mathbb{F}_q[[x^{a_1}, x^{a_2}, \cdots, x^{a_k}]]$$

is always a $\text{CK}^* - n$ domain (if $a_1 > 1$).
But this approach is death on a hotplate when it comes to trying to verify that there is a CK*-n domain for all $n \geq 3$.
We now show a theorem from Anderson and Mott that shows that CK domains are very specialized.

Theorem

For an integral domain R, the following conditions are equivalent.

1. *R is a CK-domain.*
2. *R is a one-dimensional semilocal domain with R/M finite for each nonprincipal maximal ideal M of R, $\overline{R}$ is a finitely generated R-module (equivalently, $(R : \overline{R}) \neq 0$), and $|Max(R)| = |Max(\overline{R})|$.*
3. *$\overline{R}$ is a semilocal PID, $|Max(R)| = |Max(\overline{R})|$, $\overline{R}$ is a finitely generated R-module, and if M is a nonprincipal maximal ideal of R, then R/M is finite.*

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If R is an atomic domain with less than three irreducibles, then R is a UFD (PID).

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Proposition

There is a non-trivial CK-3 domain.*

Although there are many non-isomorphic CK-3* domains, they all share the same monoid factorization structure (but in general how many of them are there?).

Here is an old observation that is key to the study of the structure of CK-domains:

Theorem

Let R be a CK domain with maximal ideals $\mathfrak{M}_1, \cdots, \mathfrak{M}_n$. Then the irreducibles of R that are contained in $\mathfrak{M}_i$ are precisely the irreducibles of $R_{\mathfrak{M}_i}$.

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Corollary

Let R be a CK domain with maximal ideals $\mathfrak{M}_1, \cdots, \mathfrak{M}_n$. Suppose that we have the factorization

$$a_1 \cdots a_n = b_1 \cdots b_n$$

with $a_i, b_i \in \mathfrak{M}_i$. Then a_i and b_i are associates.

This means all "nonuniqueness" of factorizations are contained in the distinct maximal ideals. We now look at constructing CK-n* domains.

Theorem

Let d be a square-free integer and $\mathbb{Z}[\omega]$ the ring of integers of $\mathbb{Q}[\sqrt{d}]$. If p is an inert prime in $\mathbb{Z}[\omega]$, then $R = \mathbb{Z}[p\omega]_{(p,p\omega)}$ is a CK^*-$(p+1)$ domain.

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Theorem

Let $S = ((p, pq\omega) \cup (q, pq\omega))^c$ and let $R = \mathbb{Z}[pq\omega]_S$, where p, q are distinct inert primes in $\mathbb{Z}[\omega]$. Then R is a CK^*-$((p+1)+(q+1))$ domain.

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Conjecture

Every even integer greater than six can be expressed as the sum of two distinct primes.

Theorem

If every even integer greater than six can be expressed as the sum of two distinct primes, then there exists a CK*-n domain for every positive integer $n \geq 3$.

Let n be an arbitrary positive integer. The power series construction described allows one to construct CK* domains for every $n < 11$, so we only consider the case where $n \geq 11$. The case of $n = 11$ will be discussed last. This allows us to apply the modified Goldbach Conjecture. If n is even, then $n - 2$ is even and by the modified Goldbach Conjecture, $n - 2 = p + q$ for some distinct primes p and q. We know that there must exist some quadratic ring of integers, say $\mathbb{Z}[\omega]$, where p and q are inert primes. Then $R = \mathbb{Z}[pq\omega]_{((p,pq\omega)\cup(q,pq\omega))^c}$ is a CK*-domain containing $(p+1)+(q+1) = n$ distinct irreducibles. For any odd integer $n = 2k+1$ for some $k > 0$, first note that if we can find a quadratic ring of integers where 2 is an inert prime, then we can

construct a CK*-3 domain by localizing at the maximal ideal lying over 2. So consider $n-5=2k-4$ which is an even integer. By the modified Goldbach Conjecture, we must have $2k-4=p+q$ where p and q are distinct odd primes. We also know there exists a quadratic ring of integers such that $2, p$ and q are all inert primes, say $\mathbb{Z}[\omega]$. Then $\mathbb{Z}[2pq\omega]_{((2,2pq\omega)\cup(p,2pq\omega)\cup(q,2pq\omega))^c}$ is a CK*-domain and contains
$(2+1)+(p+1)+(q+1)=5+(2k-4)=2k+1=n$
irreducible elements as desired. For the case of $n=11$, we mimic the odd case, only now we use the primes 2 and 7. For an explicit quadratic ring of integers, consider $\mathbb{Z}[(\frac{1+\sqrt{5}}{2})]=\mathbb{Z}[\omega]$. Then $\mathbb{Z}[2\cdot 7\omega]_{((2,2\cdot 7\omega))\cup(7,2\cdot 7\omega))^c}$ is a CK*-11 domain.

This problem has been solved in general by P. Clark, S. Gosavi, and P. Pollack. The tecnique is essentially the same racheted up to more general number fields. This gets around the duplication obstruction.

Other directions.

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For example, any UFD is a generalized CK-domain as is any CK-domain.

Here is another more general definition that captures the spirit of CK domains on a more global level.

Definition

We say that R is almost CK if R is atomic and given any irreducible $\alpha \in R$*,* α *has only one prime minimal over it and if* $\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_n$ *with each* α_i, β_i *having the same minimal prime then* α_i *and* β_i *are associates.*

Of course a remaining question for these domains (and CK domains) is the question of the general structure of the quasilocal case.

Thank you for having me!!