Implicitization of tensor product surfaces in the presence of a generic set of basepoints

Eliana Duarte

University of Illinois Urbana-Champaign

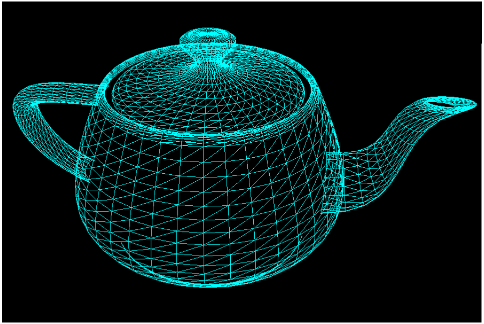
April 16, 2016

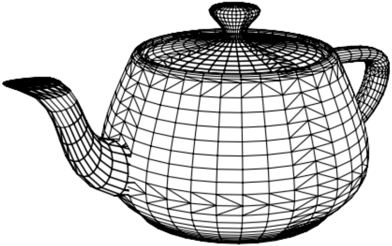
1. Motivation

2. Background and example

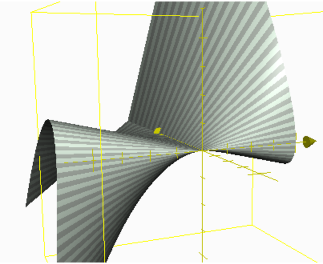
3. Main theorem and corollary

Algebraic surfaces in computer graphics





Different ways to describe algebraic surfaces



$$r(t, u) = (u, t^2, t^2 u + tu)$$

$$X^2 Y - X^2 Y^2 + 2XYZ - Z^2 = 0$$

Parametric: For any pair $(s, t) \in \mathbb{R}^2$ you associate a point $r(s, t) \in \mathbb{R}^3$ that lies in the surface.

Implicit: The surface is described as the set of all points $(x, y, z) \in \mathbb{R}^3$ that satisfy an equation.

Implicitization meets commutative algebra and algebraic geometry

$$\mathbb{A}^2 \to \mathbb{A}^3 \Longrightarrow \lambda : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$$

A *tensor product surface* (TPS) is the closed image of a map $\lambda : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$.

Goal

Given a generically finite rational map $\lambda : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ *with basepoints* the goal is to understand the syzygies of the defining polynomials of λ to set up faster algorithms that compute the implicit equation of $\overline{\mathrm{im}\ \lambda}$.

Notation for TPS

- $R = k[s,t,u,v]$ with grading $\deg(s,t) = (1,0)$ and $\deg(u,v) = (0,1)$.
- A map $\lambda : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ is determined by four elements $f_0, \ldots, f_3 \in R_{(a,b)}$ without common factors.
- $k[X,Y,Z,W]$ is the coordinate ring for $\mathbb{P}^3$.
- $B = \mathbb{V}(f_0, \ldots, f_3) \subset \mathbb{P}^1 \times \mathbb{P}^1$ is the set of basepoints of U.

Example

$$\left.\begin{aligned} f_0 &= s^2 v \\ f_1 &= stv \\ f_2 &= stu \\ f_3 &= t^2 u \end{aligned}\right. \quad \in R_{(2,1)}$$

$$U := \{f_0, \ldots, f_3\}$$

U defines a rational map $\lambda_U : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$

$$[s,t,u,v] \mapsto [f_0, f_1, f_2, f_3].$$

$$X_U := \overline{\lambda_U} = V(XW - YZ)$$

$$I_U = \langle f_0, \ldots, f_3 \rangle \subset R$$

Rees-Algebra techniques to find implicit equations

- $I_U = \langle f_0, \ldots, f_3 \rangle \subset R$, we have a map

$$R[X, Y, Z, W] \xrightarrow{\beta} R[t]$$

$$\mathrm{Rees}_R(I_U) = R[X, Y, Z, W]/\ker\beta, \quad (\ker\beta)_0 = (H).$$

- $\mathrm{Rees}_R(I_U)$ is in general hard to study so we look at $\mathrm{Sym}_R(I_U)$

$$R[X, Y, Z, W] \xrightarrow{\alpha} \mathrm{Sym}_R(I_U)$$

$$\mathrm{Sym}_R(I_U) = R[X, Y, Z, W]/\mathrm{Syz}\ I_U.$$

- If $\lambda_U : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ has finitely many local complete intersection basepoints, then $\mathrm{Sym}_R(I_U)$ and $\mathrm{Rees}_R(I_U)$ are "the same". Thus we may find $(\ker\beta)_0$ by looking at $\mathrm{Sym}_R(I_U)$ which ultimately means studying $\mathrm{Syz}\ I_U$.

Implicit equations via syzygies

Theorem[Botbol 2011]

Let $U = \mathrm{Span}\{f_0, \ldots, f_3\} \subset R_{(a,b)}$ and $\lambda_U : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ the rational map defined by U. Then the syzygies of I_U in degree $\nu = (2a-1, b-1)$ determine a complex $\mathcal{Z}_\nu$ such that

$$\det \mathcal{Z}_\nu = H^{\deg \lambda_U}$$

where H denotes the irreducible implicit equation of X_U.

$$\mathcal{Z}_\nu : \qquad 0 \longrightarrow (\mathcal{Z}_2)_\nu \xrightarrow{d_2} (\mathcal{Z}_1)_\nu \xrightarrow{d_1} (\mathcal{Z}_0)_\nu \longrightarrow 0$$

$$\det \mathcal{Z}_\nu = \frac{\det d_1}{\det d_2}$$

Points in $\mathbb{P}^1 \times \mathbb{P}^1$

STRATEGY: To understand λ_U when U has basepoints we study the ideals of R corresponding to points in $\mathbb{P}^1 \times \mathbb{P}^1$.

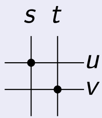
- $P = A \times C \in \mathbb{P}^1 \times \mathbb{P}^1$
- Take $h \in R_{(1,0)}$, $h(A) = 0$, and $q \in R_{(0,1)}$, $q(C) = 0$. Then

$$I_P = \langle h, q \rangle$$

- If $B = \{P_1, \ldots, P_r\}$, then

$$I_B = \bigcap_{i=1}^{r} I_{P_i}$$

Example



$$I_B = \langle s, u \rangle \cap \langle t, v \rangle$$

TPS with a generic set of basepoints

- The bigraded Hilbert function of B is defined by

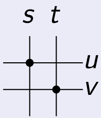
$$H_B(i,j) = \dim_k R_{(i,j)} - \dim_k (I_B)_{(i,j)}$$

- B is said to be generic if

$$H_B(i,j) = \min\{(i+1)(j+1), r\}$$

- Take U to be a generic 4-dimensional vector subspace of $(I_B)_{(a,1)}$

Example



$H_X =$

	0	1	2	3
0	1	2	2	2
1	2	2	2	2
2	2	2	2	2
3	2	2	2	2

$$I_B = \langle s, u \rangle \cap \langle t, v \rangle$$

Generic set of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

$$\begin{aligned} f_0 &= s^2 v \\ f_1 &= stv \\ f_2 &= stu \\ f_3 &= t^2 u \end{aligned} \quad \in (I_B)_{(2,1)}$$

$$U = \{f_0, f_1, f_2, f_3\}$$

Example with two generic basepoints

$a = 2, r = 2, U = \{f_0, f_1, f_2, f_3\}$

$$\begin{array}{ll} f_0 = s^2 v & f_2 = stu \\ f_1 = stv & f_3 = t^2 u \end{array}$$

- Using Bótbol's results, find a basis for the syzygies in bidegree $(2a-1, b-1) = (3,0)$, and use it to set up the complex $\mathcal{Z}_\nu$.
-

$$\mathrm{Syz}\ I_U \quad = \quad \begin{pmatrix} -t & 0 & 0 \\ s & 0 & -u \\ 0 & t & v \\ 0 & -s & 0 \end{pmatrix}$$

- We write $-t \cdot X + s \cdot Y = 0$ and use $\{s^2, st, t^2\}$ to bump up, e.g

$$-s^2 t \cdot X + s^3 \cdot Y = 0$$

Example with two generic basepoints

- Fix $R_{(3,0)} = \{s^3, s^2t, st^2, t^3\}$ and write the coefficient vectors of the syzygies w.r.t this basis.
- $$d_1 = \begin{pmatrix} Y & 0 & 0 & W & 0 & 0 \\ -X & Y & 0 & -Z & W & 0 \\ 0 & -X & Y & 0 & -Z & W \\ 0 & 0 & -X & 0 & 0 & -Z \end{pmatrix}$$
  $$-s^2t \cdot X + s^3 \cdot Y = 0$$
- Proceed in the same way for all syzygies in bidegree $(3, 0)$ to obtain the rest of the columns of M.
- $d_2 = \ker d_1$

- We obtain the complex

$$\mathcal{Z}_\nu : 0 \longrightarrow (\mathcal{Z}_2)_\nu \xrightarrow{\begin{pmatrix} W & 0 \\ -Z & W \\ 0 & -Z \\ -Y & 0 \\ X & -Y \\ 0 & X \end{pmatrix}} (\mathcal{Z}_1)_\nu \xrightarrow{\begin{pmatrix} Y & 0 & 0 & W & 0 & 0 \\ -X & Y & 0 & -Z & W & 0 \\ 0 & -X & Y & 0 & -Z & W \\ 0 & 0 & -X & 0 & 0 & -Z \end{pmatrix}} (\mathcal{Z}_0)_\nu$$

- $d_2 = \ker d_1$, $\dim \ker d_1 = 2 = \#$ of basepoints.

$$\det \mathcal{Z}_\nu = \frac{\begin{vmatrix} Y & 0 & 0 & W \\ -X & Y & 0 & -Z \\ 0 & -X & Y & 0 \\ 0 & 0 & -X & 0 \end{vmatrix}}{\begin{vmatrix} X & -Y \\ 0 & X \end{vmatrix}} = \frac{-X^2YZ + X^3W}{X^2} = XW - YZ$$

The Main theorem tells us that for any bidegree $(a, 1)$ and any set of r generic basepoints, then d_1 is always determined by two syzygies.

Main Theorem (-):

Let $(I_B)_{(a,1)}$ be the k-vector space of forms of bidegree $(a,1)$ that vanish at a generic set B of $r = 2k+1$ points in $\mathbb{P}^1 \times \mathbb{P}^1$. Take U to be a generic 4-dimensional vector subspace of $(I_B)_{(a,1)}$ and $\lambda_U : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ the rational map determined by U. Then the complex $\mathcal{Z}_\nu$, $\nu = (2a-1, 0)$ is determined by two syzygies S_1, S_2 of $(f_0, \ldots, f_3)$.

- If $r = 2k$ then $\deg S_1 = \deg S_2 = (a-k, 0)$.
- If $r = 2k+1$ then $\deg S_1 = (a-k, 0)$ $\deg S_2 = (a-(k+1), 0)$.
- The syzygies S_1, S_2 span a free module.

Main Theorem (-):

Let $(I_B)_{(a,1)}$ be the k-vector space of forms of bidegree $(a,1)$ that vanish at a generic set X of $r = 2k+1$ points in $\mathbb{P}^1 \times \mathbb{P}^1$. Take U to be a generic 4-dimensional vector subspace of $(I_B)_{(a,1)}$ and $\lambda_U : \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ the rational map determined by U. Then the complex $\mathcal{Z}_\nu$, $\nu = (2a-1, 0)$ is determined by two syzygies S_1, S_2 of $(f_0, \ldots, f_3)$.

Corollary

If $B = \emptyset$ then d_1 is determined by two syzygies of $(f_0, \ldots, f_3)$ in bidegee $(a, 0)$.

Questions

What happens for (a, b), $b > 1$?
What if U is not generic?
What if the basepoints have multiplicities?