Toric matrix Schubert varieties

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Special Session on Commutative Algebra and Its Interactions with Combinatorics and Algebraic Geometry
Fargo, ND

April 16, 2016

Joint work with Karola Mészáros (Cornell University)

- Toric and Schubert varieties are important examples of orbit closures in combinatorial algebraic geometry.
- Matrix Schubert varieties were introduced by Fulton to study degeneraci loci of flagged vector bundles.
- Knutson and Miller showed that Schubert polynomials are multidegrees of matrix Schubert varieties, and, using Gröbner bases and pipe dream complexes, studied the combinatorics of Schubert polynomials and determinant ideals building up on work by Fomin-Kirillov and Bergeron-Billey.
- There is a stratification of the flag variety by Schubert varieties and thus the cohomology of flag varieties is spanned by Schubert varieties.

In the first part of this talk I give a classification of the matrix Schubert varieties that are toric (with respect to a natural torus action).

Matrices

Notation

- $M_n = n \times n$ matrices over $\mathbb{C}$.
- $GL_n(\mathbb{C}) =$ invertible matrices in M_n.
- $B_+ =$ upper triangular matrices in M_n.
- $B_- =$ lower triangular matrices in M_n.

Group action

- The multiplications XM with $X \in B_-$ and $M \in M_n$ correspond to downward row operations.
- The multiplications MY with $Y \in B_+$ and $M \in M_n$ correspond to rightward column operations.
- Let $B_- \times B_+$ act on M_n by $(X, Y) \cdot M := XMY^{-1}$.
  This is indeed an action because $(V, W) \cdot ((X, Y) \cdot M) = (VX)M(WY)^{-1} = (VX, WY) \cdot M$.

Permutation matrices

Given a permutation $\pi \in S_n$, we also denote by π its permutation matrix.

Example

$\pi = [3, 1, 4, 2] \in S_4$ corresponds to the permutation matrix

$$\pi = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \in GL_n(\mathbb{C})$$

We denote by $B_-\pi B_+$ the $B_- \times B_+$-orbit of π.

For each $M \in GL_n(\mathbb{C})$ there exists a unique $\pi \in S_n$ such that $M \in B_-\pi B_+$.

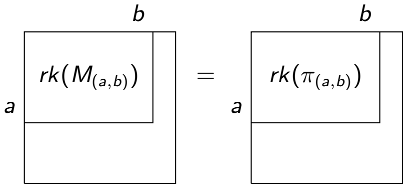
Orbits of permutation matrices

Given a permutation $\pi \in S_n$, we also denote by π its permutation matrix.

For each $M \in GL_n(\mathbb{C})$ there exists a unique $\pi \in S_n$ such that $M \in B_- \pi B_+$.

Criterion

$M \in B_- \pi B_+$ if and only if for all $(a, b) \in [n]^2$, the rank of $M_{(a,b)}$ equals the number of 1's in the NW-most $a \times b$-rectangle in π.



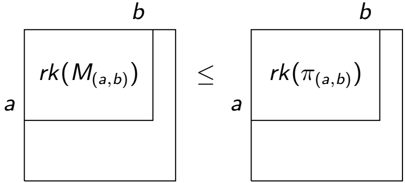
Matrix Schubert varieties

Definition

A *matrix Schubert variety* is a $(B_- \times B_+)$-orbit closure $\overline{X_\pi} := \overline{B_- \pi B_+} \subset M_n$.

Theorem (Fulton, '92)

$\overline{X_\pi}$ is an irreducible affine variety of dimension $n^2 - \ell(\pi)$ and defined as a scheme by the n^2 equations.



Matrix Schubert varieties

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$$\pi = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \in GL_n(\mathbb{C}),$$

then

$$M = \begin{bmatrix} m_{(1,1)} & m_{(1,2)} & m_{(1,3)} & m_{(1,4)} \\ m_{(2,1)} & m_{(2,2)} & m_{(2,3)} & m_{(2,4)} \\ m_{(3,1)} & m_{(3,2)} & m_{(3,3)} & m_{(3,4)} \\ m_{(4,1)} & m_{(4,2)} & m_{(4,3)} & m_{(4,4)} \end{bmatrix} \in \overline{X_{[3,1,4,2]}}$$

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Matrix Schubert varieties

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Matrix Schubert varieties

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Given $\pi = [3,1,4,2] \in S_4$ corresponding to the permutation matrix

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then

$$M = \begin{bmatrix} 0 & 1 & 2 & m_{(1,4)} \\ 0 & 3 & m_{(2,3)} & m_{(2,4)} \\ m_{(3,1)} & m_{(3,2)} & m_{(3,3)} & m_{(3,4)} \\ m_{(4,1)} & m_{(4,2)} & m_{(4,3)} & m_{(4,4)} \end{bmatrix} \in \overline{X_{[3,1,4,2]}}$$

Matrix Schubert varieties

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Given $\pi = [3, 1, 4, 2] \in S_4$ corresponding to the permutation matrix

$$\pi = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \in GL_n(\mathbb{C}),$$

then

$$M = \begin{bmatrix} 0 & 1 & 2 & m_{(1,4)} \\ 0 & 3 & 6 & m_{(2,4)} \\ m_{(3,1)} & m_{(3,2)} & m_{(3,3)} & m_{(3,4)} \\ m_{(4,1)} & m_{(4,2)} & m_{(4,3)} & m_{(4,4)} \end{bmatrix} \in \overline{X_{[3,1,4,2]}}$$

Fulton's essential set

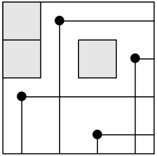
Some of the n^2 rank equations are redundant. Fulton gave a set of at most $\ell(\pi)$-many equations that define $\overline{X_\pi}$.

Definition

The *diagram $D(\pi)$* of $\pi \in S_n$ consists of the entries in the $n \times n$-matrix that remain after we cross out the entries S and E of each 1 in the permutations matrix π.

Example

$D([3,1,4,2]) = \{(1,1),(2,1),(2,3)\}$



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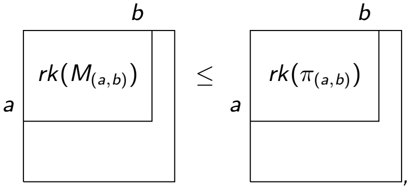
Theorem (Fulton, '92)

The ideal defining $\overline{X_\pi}$ is generated by the equations



for $(a, b) \in Ess(\pi) := \{SE \text{ corners of } D(\pi)\}$.

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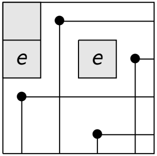
Example

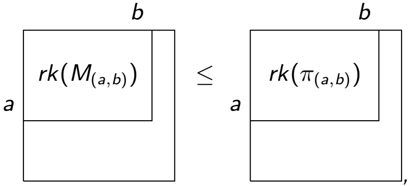


$Ess([3,1,4,2]) = \{(2,1),(2,3)\}$ and
$\overline{X_\pi} = V\left(\langle x_{(1,1)}, x_{(2,1)}, x_{(1,2)}x_{(2,3)} - x_{(1,3)}x_{(2,2)}\rangle\right)$

Fulton's essential set

Theorem (Fulton, '92)

The ideal I_π *defining* $\overline{X_\pi}$ *is generated by the equations*



for $(a,b) \in Ess(\pi) := \{SE \text{ corners of } D(\pi)\}$.

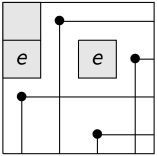
Straightforward observations

- If (a,b) is in the connected component of $(1,1)$ in $D(\pi)$ then $m_{(a,b)} = 0$.
- If (a,b) is not NW of any entry in $D(\pi)$, then $m_{(a,b)}$ is free.

Fulton's essential set

Example

Given $\pi = [3,1,4,2] \in S_4$ with diagram and essential set



then

$$M = \begin{bmatrix} 0 & ? & ? & * \\ 0 & ? & ? & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \in \overline{X_{[3,1,4,2]}}$$

here $*$ denotes a free entry

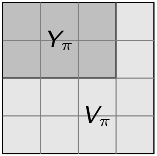
Decomposition of $\overline{X_\pi}$

Definition

Let Y_π be the variety obtained by restricting $\overline{X_\pi}$ to the entries NW of some entry of $D(\pi)$.
Let V_π be the variety obtained by restricting $\overline{X_\pi}$ to the entries not NW of any entry in $D(\pi)$.

Example

Given $\pi = [3, 1, 4, 2] \in S_4$, then Y_π and V_π have the coordinates restricted to the entries in the shading region below



Decomposition of $\overline{X_\pi}$

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Let Y_π be the variety obtained by restricting $\overline{X_\pi}$ to the entries NW of some entry of $D(\pi)$.
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Observations

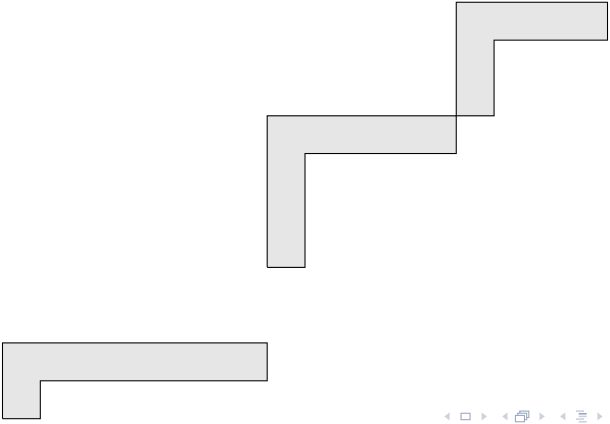
- $\overline{X_\pi} = Y_\pi \times V_\pi$
- $V_\pi = \mathbb{C}^{\text{something}}$
- $Y_\pi = V(I_\pi)$ with $I_\pi \subset \mathbb{C}[x_{(i,j)} \mid (i,j) \text{ is NW of some entry of } D(\pi)]$.
- $\dim(Y_\pi) = |NW(\pi)| - |D(\pi)|$, where

$$NW(\pi) := \{\textit{entries NW of some entry in } D(\pi)\}$$

Toric matrix Schubert ideals

Theorem (E.-Mészáros, '15)

Y_π is a toric variety (with respect to a natural torus) if and only if $NW(\pi) - D(\pi)$ consists of disjoint hooks that do not share a row or a column with each other.



Toric varieties

- $T^n =$ diagonal matrices in $GL_n(\mathbb{C}) \subset B_- \cap B_+$.
- $T^{2n} = T^n \times T^n$ acts on M_n by $(X, Y) \cdot M = XMY^{-1}$.

Definition

a normal variety X is a *toric variety* with respect to a T-torus action, if $X = \overline{T \cdot x}$ for some $x \in X$.

When is Y_π a toric variety?

- Since Y_π is an irreducible variety, if there exists an $x \in Y_\pi$ such that $\dim(\overline{T^{2n} \cdot x}) = \dim(Y_\pi)$, then Y_π is a toric variety with respect to T^{2n}.
- Since $\overline{T^{2n} \cdot x}$ is an affine toric variety, it corresponds to a cone. For $x \in Y_\pi$ is a general point, the cone is spanned by the T-weights of the action.
- $\dim(\overline{T^{2n} \cdot x})$ equals the dimension of this cone.

The cone for a general point $x \in Y_\pi$

Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n \times 0$ and $f_1, \ldots, f_n$ be the standard basis for $0 \times \mathbb{R}^n$.
The cone corresponding to $\overline{T^{2n} \cdot x}$ is spanned by the vectors $e_i - f_j$ such that $(i, j) \in NW(\pi)$ and not in the connected component of $(1, 1)$.

When is Y_π a toric variety?

Theorem (E.-Mészáros, '15)

Y_π is a toric variety (with respect to a natural torus) if and only if $NW(\pi) - D(\pi)$ consists of disjoint hooks that do not share a row or a column with each other.

Corollary

If $NW(\pi) - D(\pi)$ is a hook, then Y_π is a toric variety with respect to T^{2n}.

When is Y_π a toric variety?

Corollary

If $NW(\pi) - D(\pi)$ is a hook, then Y_π is a toric variety with respect to T^{2n}.

Proof.

Since $\dim(Y_\pi)$ equals the size of the hook, it suffices to show that the dimension of the cone equals the size of the hook. The vectors corresponding to the entries of the hook are

$$\begin{array}{|cccc|}
\hline
e_i - f_j & e_i - f_{j+c} & \cdots & e_i - f_{j+1} \\
\cline{2-4}
e_{i+1} - f_j & \multicolumn{3}{|c}{} \\
\vdots & \multicolumn{3}{|c}{} \\
e_{i+r} - f_j & \multicolumn{3}{|c}{} \\
\cline{1-1}
\end{array}$$

and these vectors are linearly independent. Thus the dimension of the cone is at least the size of the hook and so it follows that $Y_\pi = \overline{T^{2n} \cdot x}$ for a general $x \in Y_\pi$.

$\square$

Root polytopes and cone(Y_π)

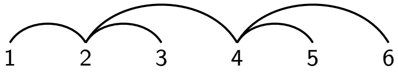
Recall the cone for a general point $x \in Y_\pi$

The cone corresponding to $\overline{T^{2n} \cdot x}$ is spanned by the vectors $e_i - f_j$ such that $(i, j) \in NW(\pi)$ and not in the connected component of $(1, 1)$.

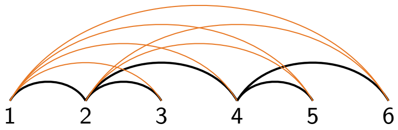
We now relate the polytope
$\Phi(Y_\pi) := convexhull\{e_i - f_j \mid (i, j) \in NW(\pi) - (conn\ component\ of\ (1, 1))\}$ to acyclic root polytopes.

Root polytopes

Given an acyclic graph $\gamma = (E(\gamma), V(\gamma))$,



take its transitive closure $\overline{\gamma}$

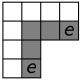


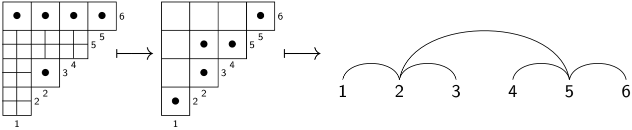
The *root polytope* of an acyclic graph γ is convexhull$\{0, e_i - e_j | (i, j) \in E(\overline{\gamma})\}$.

Root polytopes and $\Phi(Y_\pi)$

Example

Let $\pi = [1, 5, 3, 4, 2]$, then we obtain a graph γ_π associated to π by the following procedure:

$NW(\pi) =$ 



Theorem (E.-Mészáros, '15)

Suppose that $\pi(1) = 1$, then the root polytope of γ_π is obtained from $\Phi(Y_\pi) =$convexhull$\{e_i - f_j \mid (i, j) \in NW(\pi) - (\text{conn component of } (1, 1))\}$ by setting $e_i = f_j$ for all $(i, j) \in Ess(\pi)$.

Pipe dream complexes via triangulations of root polytopes

Canonical triagulations of root polytopes, defined by Mészáros, give regular triangulations of the polytopes
$\Phi(Y_\pi) = convexhull\{e_i - f_j \mid (i,j) \in NW(\pi) - (conn\ component\ of\ (1,1))\}$.

These triangulations realize pipe dream complexes when $\pi = 1\pi'$ with π' dominant.

Theorem (E.-Mészáros, 2015)

Let $\Delta_1, \ldots, \Delta_k$ be the top dimensional simplices in the canonical triangulation of $P(\gamma_\pi)$, the root polytope of γ_π, for $\pi = 1\pi'$, where π' is dominant. Then the preimages of these simplices under the map

$$\Phi(Y_\pi) \to P(\gamma_\pi),\ set\ e_i = f_j\ for\ (i,j) \in Ess(\pi)$$

are the top dimensional simplices in a triangulation of $\Phi(\mathbb{P}(Y_\pi))$ which yields geometric realization of the pipe dream complex of π.

Thank you!