Toric ideals of neural codes

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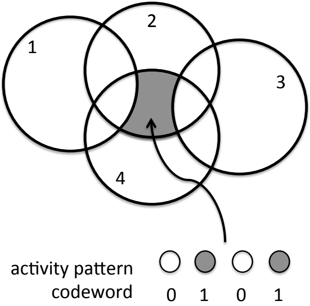
Neural activity and place fields

The freely moving rat (O'Keefe, 1979)

Receptive field code



$\mathcal{C} \subset \{0,1\}^4$

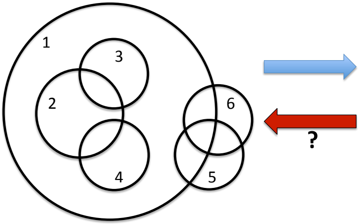
codewords:
1000 0100 0010 0001
1100 0110 1001 0011
0101 1101 0111

Let $\mathcal{U} = \{U_1, ..., U_n\}$ with $U_i \subset \mathbb{R}^2$. The neural code associated to $\mathcal{U}$ is

$$\mathcal{C}(\mathcal{U}) = \{c \in \{0,1\}^n \,|\, (\bigcap_{i \in \mathrm{supp}(c)} U_i) \setminus (\bigcup_{j \notin \mathrm{supp}(c)} U_j) \neq \emptyset\}.$$

Note: The code $\mathcal{C}$ is the data that is returned from an experiment. There is no knowledge of the sets in $\mathcal{U}$.

Drawing receptive fields



Neural code $\mathcal{C}$:

100000 000010 000001
110000 101000 100100
100010 100001 000011
111000 110100 100011

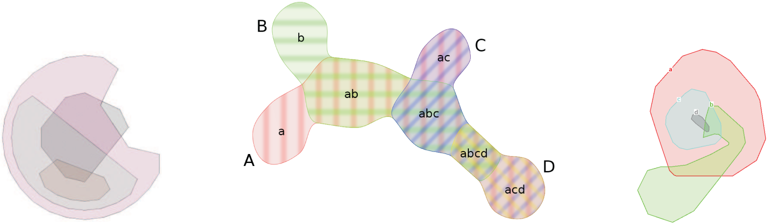
Drawing realizations: *Given a neural code, is there an algorithm to draw a realization (a Euler diagram) with convex fields?*

- **Convexity:** Not all codes are realizable with convex place fields. When is a neural code convexly realizable? (Curto–Gross–Jeffries–Morrison–Omar–Rosen–Shiu–Youngs)
- **Dimension:** What is the minimum dimension for which a convex neural code is realizable? (Rosen–Zhang, full characterization for dimension 1)

Drawing Euler diagrams

Automatically drawing Euler diagrams using “nice” shapes is quite tricky. It is a topic of current interest in the field of Information Visualization.

$$\mathcal{C} = \{0000, 1000, 0100, 1100, 1110, 1011, 1111\}$$



Images from Stapleton–Zhang–Howse–Rodgers 2013

Drawing Euler diagrams

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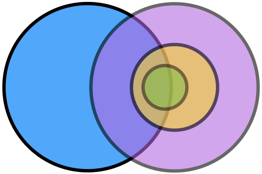
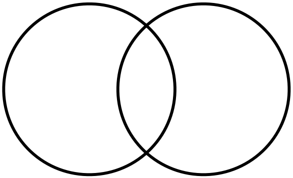


There exists an algorithm for drawing Euler diagrams with circles (Stapleton–Flower–Rodgers–Howse, 2013) for **inductively pierced** codes.

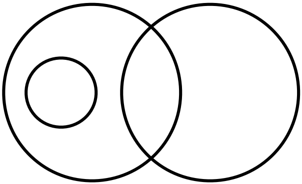
Goal: Use toric ideals to identify inductively pierced codes.

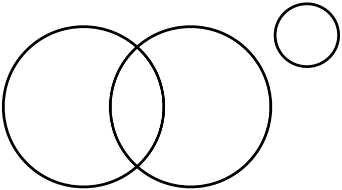
0-piercings

$D =$

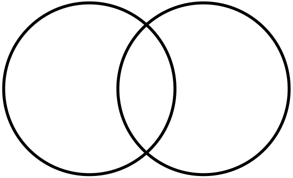


0-piercings of D:

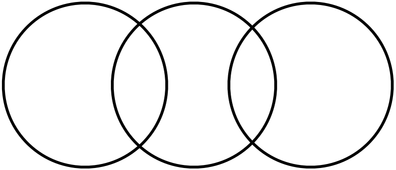


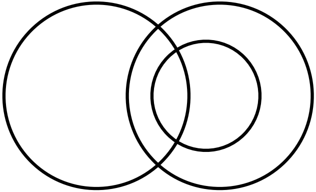


$D =$



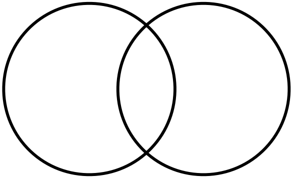
1-piercings of D:



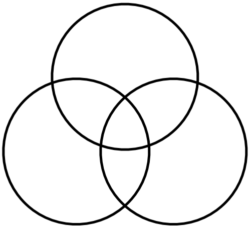


2-piercings

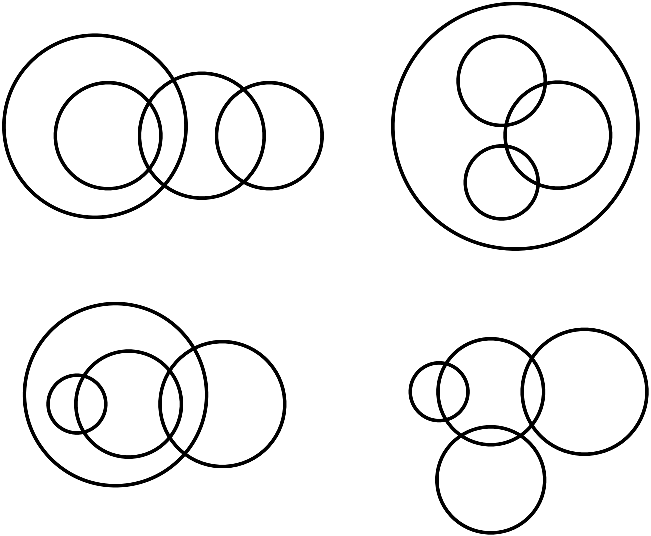
$D =$



2-piercing of D:



1-inductively pierced diagrams



k-inductively pierced

Definition

Let $\mathcal{C}$ be a neural code on n neurons. Let $\Lambda = \{\lambda_1, \ldots, \lambda_k\} \subseteq \{1, 2, \ldots, n\} = [n]$. Then $\lambda_{k+1} \in [n]$ is a *k-piercing* of Λ in $\mathcal{C}$ if there exists $\mathbf{c}^* \in \mathcal{C}$ such that

1. $\lambda_i \notin \operatorname{supp}(\mathbf{c}^*)$ for $i \leq k+1$
2. $\{\operatorname{supp}(\mathbf{c}) \ : \lambda_{k+1} \in \operatorname{supp}(\mathbf{c})\} = \{\operatorname{supp}(\mathbf{c}^*) \cup \{\lambda_{k+1}\} \cup \Lambda_i \ : \ \Lambda_i \subseteq \Lambda\}$
3. $\{\operatorname{supp}(\mathbf{c}^*) \cup \Lambda_i \ : \ \Lambda_i \in \Lambda\} \subseteq \{\operatorname{supp}(\mathbf{c}) \ : \ \mathbf{c} \in \mathcal{C}\}$.

Definition

A neural code $\mathcal{C}$ is *k-inductively pierced* if $\mathcal{C}$ has a $0, 1, \ldots,$ or k piercing λ and $\mathcal{C} - \lambda = \{(c_1, \ldots, c_{\lambda-1}, \hat{c}_\lambda, c_{\lambda+1}, \ldots, c_n) \ : (c_1, \ldots, c_n) \in \mathcal{C}\}$ is k-inductively pierced.

We can explore k-inductively pierced codes using neural ideals and their canonical form (Curto–Ikskov–Veliz-Cuba–Youngs 2013) or **toric ideals of neural codes**.

Toric ideals of neural codes

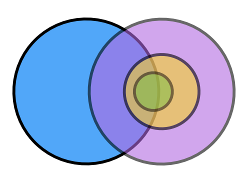
- Let $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_m\}$ be a neural code on n neurons.
- Let

$$\phi_{\mathcal{C}} : \mathbb{K}[p_c \mid c \in \mathcal{C}] \to \mathbb{K}[x_i \mid i \in \{1, \ldots, n\}]$$

$$p_c \mapsto \prod_{i \in \mathrm{supp}(c)} x_i.$$

- The toric ideal of the neural code $\mathcal{C}$ is $I_{\mathcal{C}} := \ker \phi_{\mathcal{C}}$.

Example



$$\mathcal{C} = \{1000, 0100, 1100, 1110, 1011, 1111\}$$

$$I_{\mathcal{C}} = \langle p_{0110}p_{1011} - p_{1111}, p_{1000}p_{0100} - p_{1100} \rangle$$

Detecting 0-inductively pierced codes

Theorem (Gross-Obatake-Youngs)

Let $\mathcal{C}$ be well-formed (think: convexly realizable in dimension 2). Then $I_{\mathcal{C}} = \langle 0 \rangle$ if and only if the neural code $\mathcal{C}$ is 0-inductively pierced.

Proof.

- $(\Rightarrow)$ $I_{\mathcal{C}} = \langle 0 \rangle \Rightarrow$ realizations of $\mathcal{C}$ have no crossings $\Rightarrow$ 0-inductively pierced.
- $(\Leftarrow)$ Prove by induction by using theory on toric ideals of hypergraphs (Petrović–Stasi 2014, Petrović–Gross 2013).

$\square$

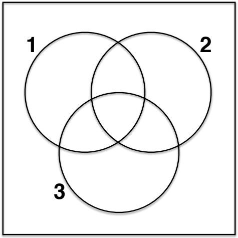
Necessary condition for 1-inductively pierced codes

Theorem (Gross-Obatake-Youngs)

Let $\mathcal{C}$ be well-formed. If the neural code $\mathcal{C}$ is 1-inductively pierced, then the toric ideal $I_{\mathcal{C}}$ is generated by quadratics or $I_{\mathcal{C}} = \langle 0 \rangle$.

Fact

Converse is not true!



- $\mathcal{C} = \{100, 010, 001, 110, 101, 011, 111\}$
- $\mathcal{C}$ is not 1-inductively pierced
- $I_{\mathcal{C}} = \langle p_{111} - p_{110}p_{001},\ p_{110} - p_{100}p_{010},\ p_{101} - p_{100}p_{001},\ p_{011} - p_{010}p_{001} \rangle$

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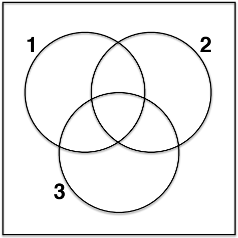

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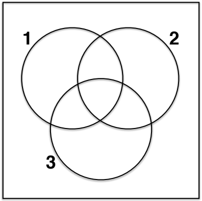
Cubic signatures of 2-piercings



- $\mathcal{C} = \{100, 010, 001, 110, 101, 011, 111\}$
- Notice: $p_{111} - p_{100}p_{010}p_{001} \in I_{\mathcal{C}}$.

Theorem (Gross-Obatake-Youngs)

Let $\mathcal{C}$ be well-formed. If $\mathcal{C}$ has a realization that contain three circles that intersect as in the figure above, then $I_{\mathcal{C}}$ contains a cubic of the form $p_{111z}p_{000z}p_{000z} - p_{100z}p_{010z}p_{001z}$ or $p_{1110\ldots0} - p_{1000\ldots0}p_{0100\ldots0}p_{0010\ldots0}$

Using Gröbner bases

Proposition (Gross-Obatake-Youngs)

A neural code $\mathcal{C}$ on 3 neurons is 1-inductively pierced if and only if the Gröbner basis of $I_{\mathcal{C}}$ with respect to the term order determined by the weight vector $[0, 0, 0, 1, 1, 1, 0]$ and GRevLex contains only binomials of degree 2 or less.

Conjecture

For all n, there exists a term order such that a code is 1-inductively pierced if and only if the Gröbner basis contains only binomials of degree 2 or less.

Thank you!