Conductor ideals of affine monoids and K-theory

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Outline

- Frobenius number of a numerical semigroup
- Affine monoid, normalization, seminormalization
- Conductor ideals & gaps in affine monoids
- Crash course in K -theory
- Affine monoid rings and their K -theory
- Nilpotence of higher K -theory of toric varieties
- Conjecture

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Huge existing literature – *Postage Stamp Problem*, *Coin Problem*, *McNugget Problem (special case)*, *Arnold Conjecture* (on asymptotics of $g(a_1, \ldots, a_n)$), etc

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REMARK. $F \cap \mathrm{sn}(M) = \mathrm{sn}(F \cap M)$ for every face $F \subset C(M)$

FACT. $\bar{M} \cap \mathrm{int}\, C(M) = \mathrm{sn}(M) \cap \mathrm{int}\, C(M)$

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The conductor ideal of an affine monoid M is

$$\mathrm{c}_{\bar{M}/M} := \{x \in \bar{M} \mid x + \bar{M} \subset M\} \subset M$$

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Proof. Let $\bar{M}$ is module finite over M . Let $\{x_1 - y_1, \ldots, x_n - y_n\} \subset \mathrm{gp}(M)$ be a generating set $x_i, y_i \in M$. Then $y_1 + \cdots + y_n \in \mathrm{c}_{\bar{M}/M}$. $\square$

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(Katthän, 2015)

$$\bar{M} \setminus M = \bigcup_{j=1}^{l} (q_j + \mathrm{gp}(M \cap F)) \cap C(M),$$

where the F_j are faces of the cone $C(M)$ and $q_j \in \bar{M}$

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where

$$g = (\sum_{i=1}^{d+1} d_i v_i)/2 - \sum_{i=1}^{d+1} v_i$$

d_i being the order of $\mathbb{Z}^d$ modulo $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_{d+1}$

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Grothendieck's group $K_0(R)$ of a ring R measures how far the projective R-modules overall are from being free (actually, stably free, which is a certain functorial weakening of 'free'')

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Informally, these groups are syzygies between elementary transformation of invertibe matricces over R . Formally, they are higher homotopy groups of a certain K -theoretical space, associated to R **(Quillen, the 1970s)**

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(Cortiñas, Haesemayer, Walker, Weibel, announced in 2016) The condition $\mathbb{Q} \subset R$ in the statement above can be dropped

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and on this module the map $M \to M$, $m \mapsto cm$, acts by dilating the M -degrees by factor c .

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It is known that $K_i(R[M])/K_i(R)$ is an R -module; this follows from the Bloch-Stienstra action of the big Witt vectors.

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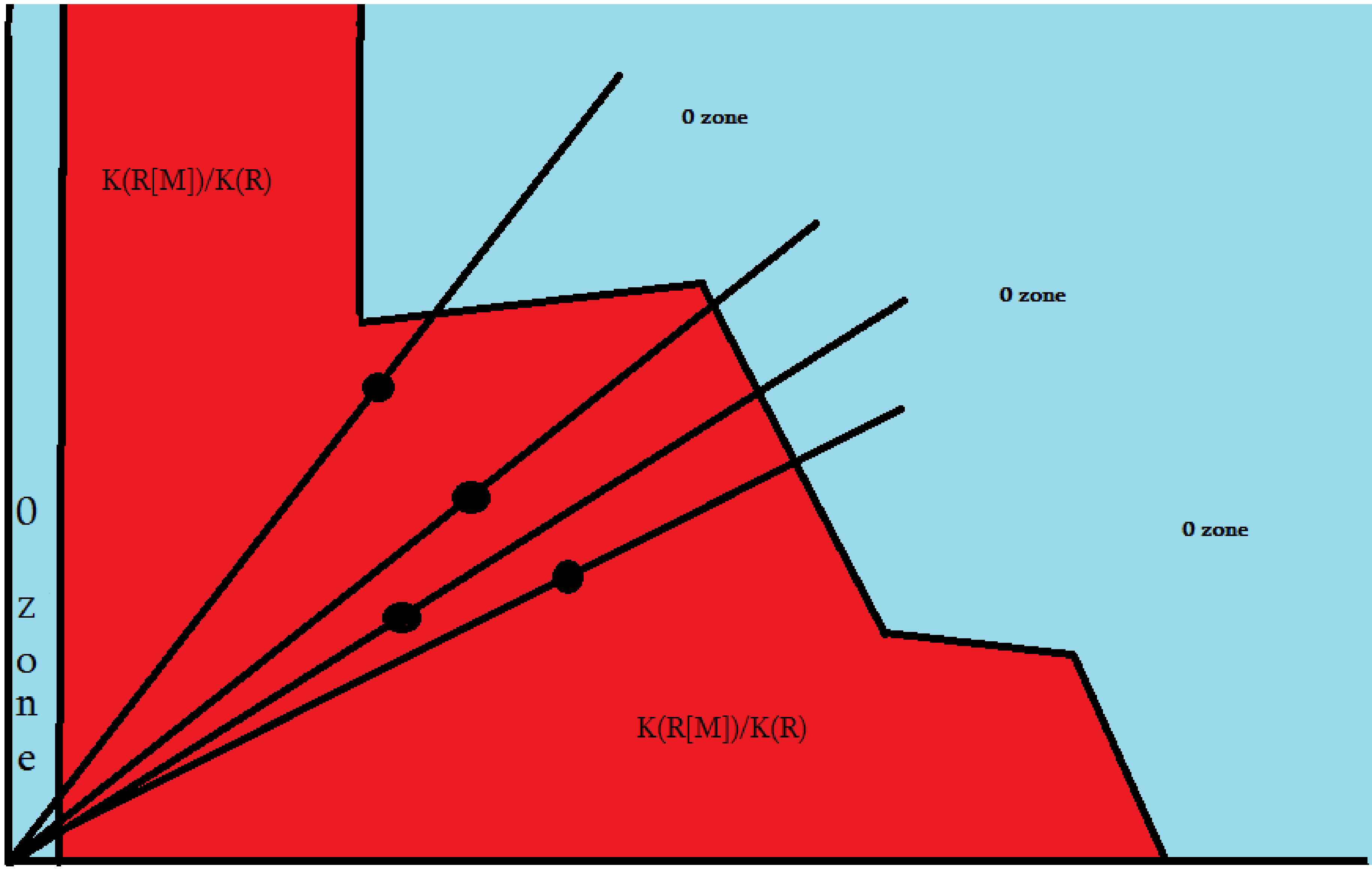

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