Symbolic powers of sums of ideals

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Problems

- Let k be a field. Let $A = k[x_1, \ldots, x_r]$ and $B = k[y_1, \ldots, y_s]$ be polynomial rings over k.
- Let $I \subseteq A$ and $J \subseteq B$ be nonzero proper homogeneous ideals.

Problem

Investigate algebraic invariants and properties of

$$(I + J)^n \text{ and } (I + J)^{(n)} \subseteq R = A \otimes_k B$$

via invariants and properties of powers of I and J.

Motivation

- **Powers of ideals** appear naturally in singularities and multiplicity theories.
- **Fiber product:** Let $X = \operatorname{Spec} A/I$ and $Y = \operatorname{Spec} B/J$. Then
$$X \times_k Y = \operatorname{Spec} R/(I + J).$$
- **Join of simplicial complexes:** Let Δ' and Δ'' be simplicial complexes on vertex sets $V = \{x_1, \ldots, x_r\}$ and $W = \{y_1, \ldots, y_s\}$, and let $\Delta = \Delta' * \Delta''$ be their join. Then
$$I_\Delta = I_{\Delta'} + I_{\Delta''}.$$
- **Hyperplane section:** $J = (y) \subseteq k[y] = B$. In this case,
$$I + J = (I, y) \subseteq k[x_1, \ldots, x_r, y].$$

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Symbolic powers of ideals

Definition

Let R be a commutative ring with identify, and let $I \subseteq R$ be a proper ideal. The n-th *symbolic power* of I is defined to be

$$I^{(n)} := R \cap \Big(\bigcap_{\mathfrak{p} \in \mathrm{Ass}_R(R/I)} I^n R_{\mathfrak{p}} \Big).$$

Example

1. If $I = \wp_1 \cap \cdots \cap \wp_s$ is the defining ideal of s points in $\mathbb{A}^n_k$ then
$$I^{(n)} = \wp_1^n \cap \cdots \cap \wp_s^n.$$
2. If I is a squarefree monomial ideal, $I = \bigcap_{\wp \in \mathrm{Ass}(R/I)} \wp$, then
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Symbolic powers of ideals

- $I^{<m>} = \left\{ f \in R \;\middle|\; \frac{\partial^{|\mathbf{a}|} f}{\partial x^{\mathbf{a}}} \in I \ \forall\ \mathbf{a} \in \mathbb{N}^n \text{ with } |\mathbf{a}| \le m-1 \right\}$.
- **Nagata, Zariski:** If $\operatorname{char} k = 0$ and I is a *radical* ideal (e.g., the defining ideal of an algebraic variety) then

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Algebraic invariants

Definition

Let R be a standard graded k-algebra, and let $\mathfrak{m}$ be its maximal homogenous ideal. Let M be a finitely generated graded R-module. Then

- $\operatorname{depth} M := \min\{i \mid H^i_{\mathfrak{m}}(M) \neq 0\}$;
- $\operatorname{reg} M := \max\{t \mid H^i_{\mathfrak{m}}(M)_{t-i} = 0 \ \forall\, i \geq 0\}$.

Grothendieck-Serre correspondence: Let $X = \operatorname{Proj} R$ and let $\widetilde{M}$ be the coherent sheaf associated to M on X. Then

$$0 \to H^0_{\mathfrak{m}}(M) \to M \to \bigoplus_{t \in \mathbb{Z}} H^0(X, \widetilde{M}(t)) \to H^1_{\mathfrak{m}}(M) \to 0$$

$$H^{i+1}_{\mathfrak{m}}(M) \cong \bigoplus_{t \in \mathbb{Z}} H^i(X, \widetilde{M}(t)) \text{ for } i > 0.$$

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Binomial expansion for symbolic powers

- $A = k[x_1, \ldots, x_r]$, $B = k[y_1, \ldots, y_s]$ are polynomial rings.
- $I \subseteq A$ and $J \subseteq B$ are nonzero proper homogeneous ideals.
- $R = A \otimes_k B = k[x_1, \ldots, x_r, y_1, \ldots, y_s]$.

Theorem (—, Trung and Trung)

For all $n \geq 1$, we have

$$(I + J)^{(n)} = \sum_{t=0}^{n} I^{(n-t)} J^{(t)}.$$

- This expansion was recently proved for *squarefree monomial ideals* by **Bocci, Cooper, Guardo, Harbourne, Janssen, Nagel, Seceleanu, Van Tuyl,** and **Vu.**

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Powers of sums of ideals by approximation

- Set $Q_p := \sum_{t=0}^{p} I^{(n-t)} J^{(t)}$. Then

$$I^{(n)} = Q_0 \subset Q_1 \subset \cdots \subset Q_n = (I+J)^{(n)}.$$

- $Q_p/Q_{p-1} = I^{(n-p)}J^{(p)}/I^{(n-p+1)}J^{(p)}$.
- There are 2 short exact sequences

$$0 \longrightarrow Q_p/Q_{p-1} \longrightarrow R/Q_{p-1} \longrightarrow R/Q_p \longrightarrow 0.$$

$$0 \longrightarrow Q_p/Q_{p-1} \longrightarrow R/I^{(n-p+1)}J^{(p)} \longrightarrow R/I^{(n-p)}J^{(p)} \longrightarrow 0.$$

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Lemma (Hoa - Tâm)

1. $\operatorname{reg} R/IJ = \operatorname{reg} A/I + \operatorname{reg} B/J + 1.$
2. $\operatorname{depth} R/IJ = \operatorname{depth} A/I + \operatorname{depth} B/J + 1.$

Theorem (—, Trung and Trung)

For $n \ge 1$, we have

1. $$\operatorname{depth} R/(I+J)^{(n)} \ge \min_{i\in[1,n-1],\ j\in[1,n]} \big\{ \operatorname{depth} A/I^{(n-i)} + \operatorname{depth} B/J^{(i)} + 1,\ \operatorname{depth} A/I^{(n-j+1)} + \operatorname{depth} B/J^{(j)} \big\}.$$
2. $$\operatorname{reg} R/(I+J)^{(n)} \le \max_{i\in[1,n-1],\ j\in[1,n]} \big\{ \operatorname{reg} A/I^{(n-i)} + \operatorname{reg} B/J^{(i)} + 1,\ \operatorname{reg} A/I^{(n-j+1)} + \operatorname{reg} B/J^{(j)} \big\}.$$

Corollary

Assume that J is generated by variables. Then

1. $\operatorname{depth} R/(I+J)^{(n)} = \min_{i \le n}\{\operatorname{depth} A/I^{(i)}\} + \dim B/J$; *and*
2. $\operatorname{reg} R/(I+J)^{(n)} = \max_{i \le n}\{\operatorname{reg} A/I^{(i)} - i\} + n$.

Depth, regularity of symbolic powers by decomposition

Proposition

$$(I+J)^{(n)}/(I+J)^{(n+1)} = \bigoplus_{i+j=n} \left(I^{(i)}/I^{(i+1)} \otimes_k J^{(j)}/J^{(j+1)}\right).$$

Theorem (—, Trung and Trung)

For all $n \geq 1$, we have

1. $\operatorname{depth} \dfrac{(I+J)^{(n)}}{(I+J)^{(n+1)}} = \min_{i+j=n} \left\{ \operatorname{depth} \dfrac{I^{(i)}}{I^{(i+1)}} + \operatorname{depth} \dfrac{J^{(j)}}{J^{(j+1)}} \right\}.$
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Corollary

The following are equivalent:

1. $R/(I+J)^{(t)}$ *is Cohen-Macaulay for all* $t \le n$;
2. $(I+J)^{(n-1)}/(I+J)^{(n)}$ *is Cohen-Macaulay;*
3. $A/I^{(t)}$ *and* $B/J^{(t)}$ *are Cohen-Macaulay for all* $t \le n$;
4. $I^{(t)}/I^{(t+1)}$ *and* $J^{(t)}/J^{(t+1)}$ *are Cohen-Macaulay for all* $t \le n-1$.

Proof of the binomial expansion

How to prove the binomial expansion

$$(I+J)^{(n)} = \sum_{t=0}^{n} I^{(n-t)} J^{(t)}?$$

- Let $S_n = \sum_{t=0}^{n} I^{(n-t)} J^{(t)}$.
- $S_n \subseteq (I+J)^{(n)}$.
- Consider the short exact sequences

$$0 \longrightarrow S_{p-1}/S_p \longrightarrow R/S_p \longrightarrow R/S_{p-1} \longrightarrow 0$$

to get

$$\mathrm{Ass}_R(R/S_n) = \bigcup_{p=1}^{n} \mathrm{Ass}_R(S_{p-1}/S_p).$$

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Problem

Let M and N be nonzero finitely generated modules over A and B, respectively. Describe the associated primes of the R-module $M \otimes_k N$ in terms of the associated primes of M and N.

Theorem (—, Trung and Trung)

Let $\operatorname{Ass}_-(-)$ *and* $\operatorname{Min}_-(-)$ *denote the set of associated and minimal primes. Then*

1. $$\operatorname{Min}_R(M \otimes_k N) = \bigcup_{\mathfrak{p} \in \operatorname{Min}_A(M), \mathfrak{q} \in \operatorname{Min}_B(N)} \operatorname{Min}_R(R/\mathfrak{p} + \mathfrak{q}).$$
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Associated primes of tensor products

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Thank You!