A combinatorial approach to the symmetry of q, t-Catalan numbers

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The q, t-Catalan numbers have several (highly nontrivially) equivalent definitions that connect different fields of mathematics including commutative algebra, combinatorics, symmetric functions, representation theory, and algebraic geometry.

Definition of commutative algebraic q, t-Catalan numbers

- Let $R = \mathbb{C}[x_1, ..., x_n, y_1, ..., y_n]$.
- Let $I \subset R$ be the bi-graded ideal (non-minimally) generated by the determinants of all matrices of the form

$$\begin{pmatrix} x_1^{a_1} y_1^{b_1} & \cdots & x_1^{a_n} y_1^{b_n} \\ \vdots & \ddots & \vdots \\ x_n^{a_1} y_n^{b_1} & \cdots & x_n^{a_n} y_n^{b_n} \end{pmatrix}, \quad a_i, b_i \in \mathbb{Z}_{\geq 0}$$

Definition

The n-th (commutative algebraic) q, t-Catalan number $CA\text{-}Cat_n(q, t)$ is the q, t-Hilbert polynomial of the minimal generators for I.

Example

For $n = 3$, the ideal is minimally generated by

$$\det\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}, \det\begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}, \det\begin{pmatrix} 1 & x_1^2 & y_1 \\ 1 & x_2^2 & y_2 \\ 1 & x_3^2 & y_3 \end{pmatrix},$$

$$\det\begin{pmatrix} 1 & x_1 & y_1^2 \\ 1 & x_2 & y_2^2 \\ 1 & x_3 & y_3^2 \end{pmatrix}, \det\begin{pmatrix} 1 & y_1 & y_1^2 \\ 1 & y_2 & y_2^2 \\ 1 & y_3 & y_3^2 \end{pmatrix}.$$

Hence $CA\text{-}Cat_3(q,t) = qt + q^3 + q^2t + qt^2 + t^3$.

1			
	1		
	1	1	
			1

$CA\text{-}Cat_3(q,t)$

1															
	1														
	1	1													
	1	1	1												
	1	2	1	1											
		2	2	1	1										
		$\ddots$	3	2	1	1									
			$\ddots$	3	2	1	1								
				$\ddots$	3	2	1	1							
					$\ddots$	3	2	1	1						
						$\ddots$	3	2	1	1					
							$\ddots$	3	2	1	1				
								$\ddots$	3	2	1	1			
									$\ddots$	2	2	1	1		
											1	1	1	1	
															1

$CA\text{-}Cat_6(q,t)$

Remark

Obviously $CA\text{-}Cat_n(q,t) = CA\text{-}Cat_n(t,q)$.

Problem

Find a set of minimal generators for I.

This is very difficult, but there is a precise conjecture (by Can–L–Li–Loehr) using a combinatorial description.

Definition of q, t-Catalan numbers using Dyck paths

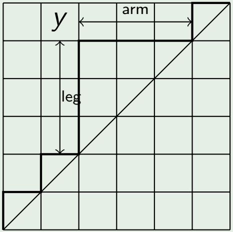
Let Γ be a Dyck path in the $n \times n$ grid.

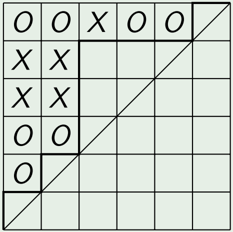
- area(Γ) $=$ the number of unit boxes between Γ and the diagonal
- A box y above γ is called

$$\begin{cases} \text{good}, & \text{if } \mathrm{arm}(y) - \mathrm{leg}(y) \in \{0, 1\}; \\ \text{bad}, & \text{otherwise.} \end{cases}$$

- dinv(Γ) $=$ the number of good boxes

Example





area$= 3$

dinv$= 7$

Definition

$$DP\text{-}Cat_n(q,t) := \sum_{\Gamma:\text{Dyck path}} q^{\mathrm{area}(\Gamma)} t^{\mathrm{dinv}(\Gamma)}$$

Theorem (Haiman, Garsia–Haglund 2001)

$$CA\text{-}Cat_n(q,t) = DP\text{-}Cat_n(q,t)$$

Corollary

$$DP\text{-}Cat_n(q,t) = DP\text{-}Cat_n(t,q)$$

Problem

Find a combinatorial proof for Corollary.

Let $\operatorname{defc}(\Gamma)$ be the number of bad boxes above Γ, i.e.,

$$\operatorname{defc}(\Gamma) := \binom{n}{2} - \operatorname{area}(\Gamma) - \operatorname{dinv}(\Gamma).$$

Definition

For $k \in \mathbb{Z}_{\geq 0}$, we define

$$DP\text{-}Cat_{n,k}(q,t) := \sum_{\operatorname{defc}(\Gamma)=k} q^{\operatorname{area}(\Gamma)} t^{\operatorname{dinv}(\Gamma)}.$$

Theorem (L-Li-Loehr 2016)

For $k \leq 9$ and all n, we have an explicit bijection for the q,t-joint symmetry:

$$DP\text{-}Cat_{n,k}(q,t) = DP\text{-}Cat_{n,k}(t,q).$$

Idea of construction

Step 1 : Construct a certain infinite sequence (and its dual) of diagrams, whose dinv are increasing by 1.
Step 2 : For each n, consider the subsequence(s) consisting of the diagrams that fit in the $n \times n$ grid as a Dyck path.

Then the Dyck paths in this subsequence(s) have the joint q, t-symmetry property.

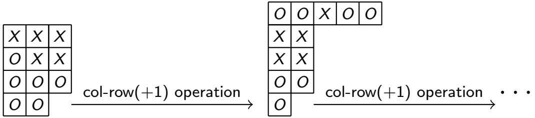
Step 3 : Decompose the set of all diagrams into such infinite sequences.

Toy Example

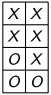
$$\begin{array}{|c|}\hline X \\ \hline X \\ \hline X \\ \hline X \\ \hline X \\ \hline O \\ \hline \end{array} \xrightarrow{\text{col-row}(+1) \text{ operation}} \begin{array}{|c|c|c|c|c|c|c|}\hline X & X & X & X & X & O & O \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline X & X \\ \hline X & X \\ \hline O & X \\ \hline O & O \\ \hline \end{array} \xrightarrow{\text{col-row}(+1) \text{ operation}} \begin{array}{|c|cccc}\hline O & \multicolumn{1}{c|}{X} & \multicolumn{1}{c|}{X} & \multicolumn{1}{c|}{O} & \multicolumn{1}{c|}{O} \\ \hline X \\ \cline{1-1} X \\ \cline{1-1} X \\ \cline{1-1} O \\ \cline{1-1} \end{array} \xrightarrow{\text{col-row}(+1) \text{ operation}} \begin{array}{|c|c|c|c|c|c|}\hline X & X & X & O & O & O \\ \hline X & X & O & O \\ \cline{1-4} \end{array}$$

Toy Example continued



Let $n = 6$. Then the corresponding subsequence consists of the following 4 Dyck paths: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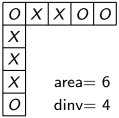
dinv= 3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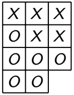
dinv= 4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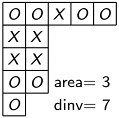
dinv= 6


area= 3
dinv= 7