Decompositions of Binomial Ideals

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Polynomial Ideals

$R = \mathbb{k}[x_1, \ldots, x_n]$ the polynomial ring over a field $\mathbb{k}$.

A monomial is a polynomial with one term, a binomial is a polynomial with at most two terms.

Monomial ideals are generated by monomials, binomial ideals are generated by binomials.

Monomial ideals:

Algebra, Combinatorics, Topology.

Toric Ideals:

Prime binomial ideals.
Algebra, Combinatorics, Geometry.

Binomial Ideals

Theorem (Eisenbud and Sturmfels, 1994)

$I \subset R$ a binomial ideal, $\mathbb{k}$ algebraically closed.

- *Geometric Statement:*
  Var(I) *is a union of toric varieties.*
- *Algebraic Statement:*
  The associated primes and primary components of I can be chosen binomial.

Why are Noetherian rings called Noetherian?

R commutative ring with 1, Noetherian (ascending chains of ideals stabilize).

A proper ideal $I \subset R$ is prime if $xy \in I$ implies $x \in I$ or $y \in I$.

I is primary if $xy \in I$ and $x^n \notin I \ \forall n \in \mathbb{N}$, implies $y \in I$.

Theorem (Lasker 1905 (special cases), Noether 1921)

Every proper ideal $I \subsetneq R$ has a decomposition as a finite intersection of primary ideals.

The radicals of the primary ideals appearing in the decomposition are the associated primes of I.

Binomial Ideals

Theorem (Eisenbud and Sturmfels, 1994)

$I \subset R$ *a binomial ideal,* $\Bbbk$ *algebraically closed.*

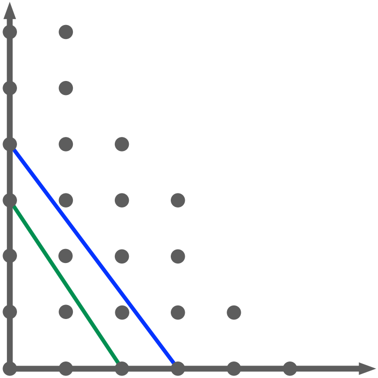
- *Geometric Statement:*
  Var(I) *is a union of toric varieties.*
- *Algebraic Statement:*
  The associated primes and primary components of I *can be chosen binomial.*
- *Combinatorial Statement:*
  The subject of this talk.

Need $\Bbbk$ algebraically closed; $\mathtt{char}(\Bbbk)$ makes a difference.
Example: In $\Bbbk[y]$, consider $I = \langle y^p - 1 \rangle$.

No hope of nice combinatorics for trinomial ideals.

There is combinatorics! (Slide of joy)

$$I = \langle x^2 - y^3, x^3 - y^4 \rangle$$
$$= \langle x - 1, y - 1 \rangle \cap (I + \langle x^4, x^3 y, x^2 y^2, x y^4, y^5 \rangle)$$



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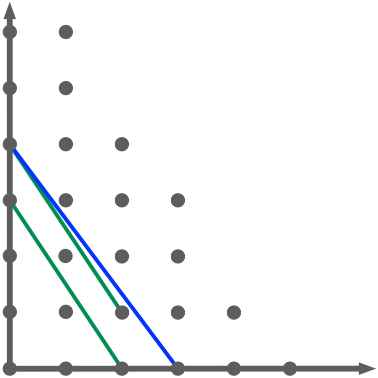

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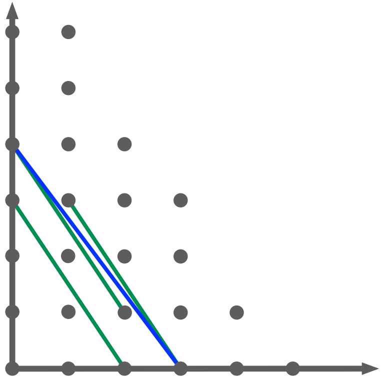

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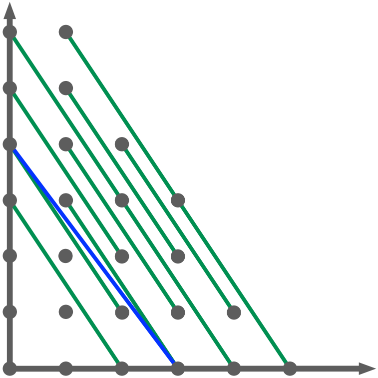

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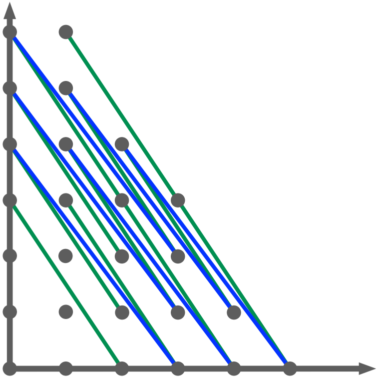

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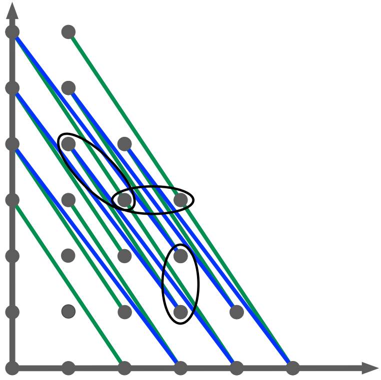

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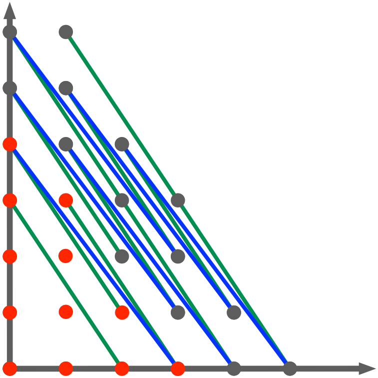

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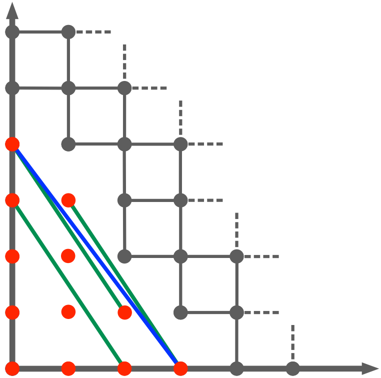

Works for binomial ideals over $\mathbb{k} = \overline{\mathbb{k}}$ with $\mathtt{char}(\mathbb{k}) = 0$.
But how to make sure we have all bounded components?

Switch gears: Lattice Ideals

If $L \subseteq \mathbb{Z}^n$ is a lattice, and $\rho : L \to \Bbbk^*$ is a group homomorphism,

$$I(\rho) = \langle x^u - \rho(u-v)x^v \mid u, v \in \mathbb{N}^n, u - v \in L \rangle \subset \Bbbk[x_1, \ldots, x_n]$$

is a lattice ideal.

Theorem (Eisenbud–Sturmfels)

A binomial ideal I is a lattice ideal iff $mb \in I$ for m monomial, b binomial $\Rightarrow b \in I$.
If $\Bbbk$ is algebraically closed, the primary decomposition of $I(\rho)$ can be explicitly determined in terms of extensions of ρ to $\mathtt{Sat}(L) = (\mathbb{Q} \otimes_{\mathbb{Z}} L) \cap \mathbb{Z}^n$.

Lattice Ideals are easy to decompose

Example

$L = \mathrm{span}_{\mathbb{Z}}\{(-1,0,3,2),(2,-3,0,1)\} \subset \mathbb{Z}^4$.

$\rho : \mathbb{Z}^4 \to \Bbbk^*$ the trivial character.

$$I(\rho) = \langle xw^2 - z^3, x^2w - y^3 \rangle.$$

$\mathrm{Sat}(L) = \mathrm{span}_{\mathbb{Z}}\{(1,-2,1,0),(0,1,-2,1)\}$ and $|\mathrm{Sat}(L)/L| = 3$

If $\mathrm{char}(\Bbbk) \neq 3$, then $I = I_1 \cap I_2 \cap I_3$, where

$$I_j = \langle yz - \omega^j xw, xz - \omega^j y^2, z^2 - \omega^{2j} yw \rangle, \quad \omega^3 = 1, \quad \omega \neq 1.$$

If $\mathrm{char}(\Bbbk) = 3$, I is primary.

What next

The good:

Relevant combinatorics: monoid congruences.

Laura, don't forget to explain what congruences are.

The not so good:

Field assumptions, computability issues.

Take a deep breath: Stop decomposing at the level of lattice ideals.

The choices:

- Finest possible
  $\rightarrow$ Mesoprimary Decomposition [Kahle-Miller]
- Coarsest possible
  $\rightarrow$ Unmixed Decomposition [Eisenbud-Sturmfels], [Ojeda-Piedra], [Eser-M]

Too many definitions

Colon ideal and saturation:
$(I : x) = \{f \mid xf \in I\}$ and $(I : x^\infty) = \{f \mid \exists \ell > 0, x^\ell f \in I\}$

I binomial ideal, m monomial $\Rightarrow (I : m), (I : m^\infty)$ binomial.

Let $\sigma \subseteq \{1, \ldots, n\}$. $I \subseteq \Bbbk[x_1, \ldots, x_n]$ is σ-cellular if $\forall i \in \sigma$, $(I : x_i) = I$, and $\forall j \notin \sigma$, $\exists \ell_j > 0$ such that $x_j^{\ell_j} \in I$.

I a σ-cellular binomial ideal.

- I is mesoprime if $I = \langle I_{\text{lat}} \rangle + \langle x_j \mid j \notin \sigma \rangle$ for some lattice ideal $I_{\text{lat}} = I_{\text{lat}} \subset \Bbbk[x_i \mid i \in \sigma]$.
- I is mesoprimary if $b \in \Bbbk[x_i \mid i \in \sigma]$ binomial, m monomial and $bm \in I \Rightarrow m \in I$ or $b \in I_{\text{lat}} = I \cap \Bbbk[x_i \mid i \in \sigma]$.
- I is unmixed if $\operatorname{Ass}(I) = \operatorname{Ass}(\langle I_{\text{lat}} \rangle + \langle x_j \mid x_j \notin \sigma \rangle)$, where $I_{\text{lat}} = I \cap \Bbbk[x_i \mid x_i \in \sigma]$.

Cellular, Mesoprimary, Unmixed

I a σ-cellular binomial ideal, mesoprime.

- I is mesoprime if $I = \langle I_{\text{lat}} \rangle + \langle x_j \mid j \notin \sigma \rangle$ for some lattice ideal $I_{\text{lat}} \subset \mathbb{k}[x_i \mid i \in \sigma]$.
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Example

$$I = \langle x^3 - 1, y(x-1), y^2 \rangle$$

cellular, unmixed, not mesoprimary, with decomposition

$$I = \langle x^3 - 1, y \rangle \cap \langle x - 1, y^2 \rangle.$$

If $\mathtt{char}(\mathbb{k}) = 3$, I is primary.
If $\mathtt{char}(\mathbb{k}) \neq 3$, the primary decomposition is
$I = \langle x - \omega, y \rangle \cap \langle x - \omega^2, y \rangle \cap \langle x - 1, y^2 \rangle; \omega^3 = 1, \omega \neq 1.$

Cellular, Mesoprimary, Unmixed

I a σ-cellular binomial ideal, mesoprime.

- I is mesoprime if $I = \langle I_{\text{lat}} \rangle + \langle x_j \mid j \notin \sigma \rangle$ for some lattice ideal $I_{\text{lat}} \subset \Bbbk[x_i \mid i \in \sigma]$.
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- I is unmixed if $\mathrm{Ass}(I) = \mathrm{Ass}(\langle I_{\text{lat}} \rangle + \langle x_j \mid x_j \notin \sigma \rangle)$, where $I_{\text{lat}} = I \cap \Bbbk[x_i \mid x_i \in \sigma]$.

Example

$I = \langle I_{\text{lat}} \rangle + \langle I_{\text{art}} \rangle$ is always mesoprimary but converse is not true. For instance

$$\langle x^2y^2 - 1, xz - yw, z^2, w^2 \rangle$$

is mesoprimary.

At last

Theorem

Decompositions of binomial ideals into

- *mesoprimary binomial ideals [Kahle-Miller]*
- *unmixed cellular binomial ideals [Eisenbud-Sturmfels] [Ojeda-Piedra] [Eser-M]*

exist over any field.

The punchline:
Now primary decomposition is easy!

But how to do it? (Handwavy slide, we are all tired)

The easy case: I is σ-cellular.

For m monomial in $\mathbb{k}[x_j \mid j \notin \sigma]$, $J_m = (I : m) \cap \mathbb{k}[x_i \mid i \in \sigma]$ is a lattice ideal.

The unmixed/mesoprimary components of I are of the form

$$((I + J_m) : \prod_{i \in \sigma} x_i^\infty) + \text{"combinatorial" monomial ideal}$$

Mesoprimary decomposition: largest possible monomial ideal.

Unmixed decomposition: smallest possible monomial ideal.

It is easy to produce mesoprimary/unmixed decompositions.
Controlling the decompositions is hard.

Slide of shame

Binomial ideals do not in general have irreducible binomial decompositions [Kahle-Miller-O'Neill].

I a binomial ideal.

- When is $\Bbbk[x]/I$ Cohen–Macaulay?
- Gorenstein?
- What are the Betti numbers of $\Bbbk[x]/I$?
- Can a (minimal) free resolution be constructed?
- Is there something like the Ishida complex?
- Ask any interesting question here...

I do not know.

The optimistic ending: An emerging area, with lots of interesting open problems!

THANK YOU!

Proof of Noether's theorem (slide of the second wind)

$I \subsetneq R$ is reducible if $I = J_1 \cap J_2$ with $J_1, J_2 \supsetneq I$.

1. Every proper ideal has an irreducible decomposition.

   If I does not have an irreducible decomposition, can produce a non-stabilizing ascending chain of ideals.

2. Irreducible ideals are primary.

   I is primary iff every $x \in R$ is either nilpotent or a nonzerodivisor modulo I.

Suppose $x \in R$ is neither nilpotent nor a nonzerodivisor mod I.

Then: $(I : x) \subset (I : x^2) \subset (I : x^3) \subset \cdots$

so $\exists N$: $(I : x^N) = (I : x^{N+1}) = \cdots$

Claim.

$$I = (I + \langle x^N \rangle) \cap (I : x^N)$$