# The smallest Borel ideal containing the product of the variables

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Let $S = k[x_1, \ldots, x_n] = k[a, b, c, \ldots]$, and let $I$ be a monomial ideal of $S$.

**Definition:** $I$ is *Borel* if it satisfies the condition:

> Let $i < j$ and let $g$ be a monomial such that $gx_j \in I$. Then $gx_i \in I$.

Changing $x_j$ to $x_i$ is called a *Borel move*.

If $m$ is a monomial, the **principal Borel-fixed ideal generated by** $m$ is the smallest Borel-fixed ideal containing $m$. We call it Borel$(m)$.

**Examples:**

$$\text{Borel}(abcd) = (a^4, a^3b, a^3c, a^2b^2, a^2bc, a^2bd,\\ a^2c^2, a^2cd, ab^3, ab^2c, ab^2d, abc^2, abcd)$$

$$\text{Borel}(x_n^k) = \mathfrak{m}^k$$

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**Answer**: Just about every invariant we care about.

Furthermore, all the invariants of Borel$(x_1 x_2 \ldots x_k)$ are really nice.

# Associated primes and primary decomposition

**Theorem (Classical):** If $B$ is Borel and $P \in \text{Ass}(B)$, then $P = P_q = (x_1, \ldots, x_q)$ for some $q$.

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**Theorem:** Suppose $B =$ Borel($m$) is principal Borel. Then

$$B = \bigcap (x_1, \ldots, x_q)^k,$$

where $x_q$ occurs in the "$k^{\text{th}}$ position" in $m$. For example,

Borel($abcd$) $= (a) \cap (a, b)^2 \cap (a, b, c)^3 \cap (a, b, c, d)^4$.

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The associated primes of Borel$(x_1 x_2 \ldots x_k)$ form a saturated chain.

# Hilbert functions and Betti numbers

Suppose that $S = k[a, b, c, d, e]$ and $abc \in B$ is a (classical) monomial generator.

Then $abc$ contributes $\{a^2bc, ab^2c, abc^2, abcd, abce\}$ to the degree-four part of $B$.

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The contribution of each generator $\mu$ to higher degrees depends only on its last variable.

Put $w_i(B) = \#\{\mu : \max(\mu) = x_i\}$.

If $B$ is a Borel ideal generated entirely in degree $d$, we have:

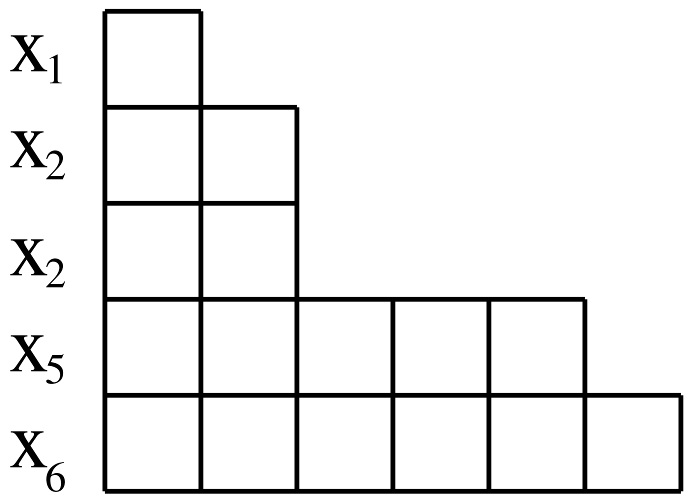
$$\mathsf{HS}(B) = \sum w_i(B) \frac{t^d}{(1-t)^{n-i+1}}.$$

Furthermore, the graded Betti numbers of $B$ are

$$\beta_{j,j+d}(B) = \sum w_i(B) \binom{i-1}{j}.$$

So we actually want to compute $w_i(B)$.

To compute the $w_i(B)$ for $B = \text{Borel}(m)$, build the Catalan diagram of shape $m$:




(Here, $m = ab^2ef = x_1 x_2^2 x_5 x_6$.)

To compute the $w_i(B)$ for Borel($m$), build the Catalan diagram of shape $m$:

| | | | | | | |
|---|---|---|---|---|---|---|
| $x_1$ | 1 | | | | | |
| $x_2$ | 1 | 1 | | | | |
| $x_2$ | 1 | 2 | | | | |
| $x_5$ | 1 | 3 | 3 | 3 | 3 | |
| $x_6$ | 1 | 4 | 7 | 10 | 13 | 13 |

...and fill it in like Catalan's triangle.

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To compute the Betti numbers of Borel($m$), build the Catalan diagram of shape $m$:

| | | | | | | |
|---|---|---|---|---|---|---|
| $x_1$ | 1 | | | | | |
| $x_2$ | 1 | 1 | | | | |
| $x_2$ | 1 | 2 | | | | |
| $x_5$ | 1 | 3 | 3 | 3 | 3 | |
| $x_6$ | 1 | 4 | 7 | 10 | 13 | 13 |

Then plug in $t+1$ to the generating function on the last row.

$$g(t) = 1 + 4t + 7t^2 + 10t^3 + 13t^4 + 13t^5$$
$$g(t+1) = 48 + 165t + 245t^2 + 192t^3 + 78t^4 + 13t^5$$

betti res module borel monomialIdeal($ab^2ef$) :

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| total: | 48 | 165 | 245 | 192 | 78 | 13 |
| 5: | 48 | 165 | 245 | 192 | 78 | 13 |

$I = (a,b,c,d)^3$:

| | | | |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 1 | 2 | 3 | 4 |
| 1 | 3 | 6 | 10 |

$$g(t) = 1 + 3t + 6t^2 + 10t^3$$
$$g(t+1) = 20 + 45t + 36t^2 + 10t^3$$

betti res borel monomialIdeal($d^3$) :

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| total: | 1 | 20 | 45 | 36 | 10 |
| 0: | 1 | . | . | . | . |
| 1: | . | . | . | . | . |
| 2: | . | 20 | 45 | 36 | 10 |

$I = \text{Borel}(abcd)$:

| | | | |
|---|---|---|---|
| 1 | | | |
| 1 | 1 | | |
| 1 | 2 | 2 | |
| 1 | 3 | 5 | 5 |

$$g(t) = 1 + 3t + 5t^2 + 5t^3$$
$$g(t+1) = 14 + 28t + 20t^2 + 5t^3$$

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betti res borel monomialIdeal(a*b*c*d) :
                    0   1   2   3   4
            total:  1  14  28  20   5
                0:  1   .   .   .   .
                1:  .   .   .   .   .
                2:  .   .   .   .   .
                3:  .  14  28  20   5
```

# Boij-Söderberg decompositions

Let $\beta(B)$ and $\beta(S/B)$ stand for the Betti diagrams. For example, if $B =$ Borel($abcd$), we have

$$\beta(B) = \begin{pmatrix} . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 14 & 28 & 20 & 5 \end{pmatrix}$$

$$\beta(S/B) = \begin{pmatrix} 1 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & 14 & 28 & 20 & 5 \end{pmatrix}$$

The Boij-Söderberg theorems say that these are positive linear combinations of the Betti diagrams of pure Cohen-Macaulay modules.

Let $B$ be a Borel ideal, generated in degree $d$. Then the Boij-Söderberg decompostion of $B$ is given by the $w_i(B)$:

$$\beta(B) = \sum w_i(B)\beta(\text{Borel}(x_i^d)).$$

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$$\begin{pmatrix} . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 14 & 28 & 20 & 5 \end{pmatrix} = 1 \begin{pmatrix} . \\ . \\ . \\ . \\ 1 \end{pmatrix} + 3 \begin{pmatrix} . & . \\ . & . \\ . & . \\ . & . \\ 1 & 1 \end{pmatrix}$$

$$+ 5 \begin{pmatrix} . & . & . \\ . & . & . \\ . & . & . \\ . & . & . \\ 1 & 2 & 1 \end{pmatrix} + 5 \begin{pmatrix} . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 1 & 3 & 3 & 1 \end{pmatrix}$$

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$$\beta(B) = \sum w_i(B)\beta(\mathrm{Borel}(x_i^d)).$$

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 14 & 28 & 20 & 5 \end{pmatrix} = 1 \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} + 3 \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & 1 \end{pmatrix}$$
$$+ 5 \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & 2 & 1 \end{pmatrix} + 5 \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 3 & 3 & 1 \end{pmatrix}$$

The situation with $S/B$ is more complicated.

When the dust clears, we get

$$\beta(S/B) = \sum \left( \frac{w_i(B)}{w_i(\mathfrak{m}_n)^d} - \frac{w_{i+1}(B)}{w_{i+1}(\mathfrak{m}_n)^d} \right) \beta(S/\mathfrak{m}_i^d),$$

where $B$ is generated in degree $d$, $\mathfrak{m}_i = (x_1, \ldots x_i)$, and $n$ is sufficiently large.

When more dust clears, $S/\operatorname{Borel}(x_1x_2x_3)$ lies at the centroid of its Boij-Söderberg face:

$$\begin{pmatrix} 1 & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & 5 & 6 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & . \\ . & . \\ . & . \\ . & 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 & . & . \\ . & . & . \\ . & . & . \\ . & 4 & 3 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & 10 & 15 & 6 \end{pmatrix}$$

When more dust clears, $S/\operatorname{Borel}(x_1x_2x_3x_4)$ lies at the centroid of its Boij-Söderberg face:

$$\begin{pmatrix} 1 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & 14 & 28 & 20 & 5 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & . \\ . & . \\ . & . \\ . & . \\ . & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & . & . \\ . & . & . \\ . & . & . \\ . & . & . \\ . & 5 & 4 \end{pmatrix}$$
$$+ \frac{1}{4}\begin{pmatrix} 1 & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & 15 & 24 & 10 \end{pmatrix}$$
$$+ \frac{1}{4}\begin{pmatrix} 1 & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & 35 & 84 & 70 & 20 \end{pmatrix}$$

When more dust clears, $S/\operatorname{Borel}(x_1x_2\ldots x_n)$ lies at the centroid of its Boij-Söderberg face:

$$\beta\left(\frac{S}{\operatorname{Borel}(x_1x_2\ldots x_n)}\right) = \frac{\sum_{i=1}^{n} \beta\left(\frac{S}{\mathfrak{m}_i^n}\right)}{n}$$