The Greedy Basis Equals the Theta Basis A Rank Two Haiku

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http://math.umn.edu/∼musiker/Haiku.pdf

Outline

1. Introduction to Cluster Algebras
2. Greedy Bases
3. Theta Bases in Rank 2
4. Sketch of their Equivalence

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Introduction to Cluster Algebras

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Cluster algebras are a certain class of commutative rings which have a distinguished set of generators that are grouped into overlapping subsets, called clusters, each having the same cardinality.

What is a Cluster Algebra?

Definition (Sergey Fomin and Andrei Zelevinsky 2001) A cluster algebra $\mathcal{A}$ (of geometric type) is a subalgebra of $k(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ constructed cluster by cluster by certain exchange relations.

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The set of all such generators are known as Cluster Variables, and the initial pattern of exchange relations (described as a *valued* quiver, i.e. a directed graph, or as a skew-symmetrizable matrix) determines the Seed.

Relations:

Induced by the Binomial Exchange Relations.

Point of view of Cluster Algebras III (Berenstein-Fomin-Zelevinsky):
$\mathcal{A}$ is generated by $x_1, x_2, \ldots, x_n, x'_1, x'_2, \ldots, x'_n$ where the standard monomials in this alphabet (i.e. x_i and x'_i forbidden from being in the same monomial) are a $\mathbb{Z}$-basis for $\mathcal{A}$.

The polynomials

$$x_i x'_i - \prod x_{\gamma_j}^{d_j^+} - \prod x_{\gamma_j}^{d_j^-}$$

generate the ideal I of relations.

Form a Gröbner basis for I assuming a term order where the x'_i's are higher degree than the x_i's.

Example: Rank 2 Cluster Algebras

Let $B = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}$, $b, c \in \mathbb{Z}_{>0}$. $(\{x_1, x_2\}, B)$ is a seed for a cluster algebra $\mathcal{A}(b, c)$ of rank 2.

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$$\mu_1(B) = \mu_2(B) = -B \text{ and } x_1 x_1' = x_2^c + 1, \quad x_2 x_2' = 1 + x_1^b.$$

Thus the cluster variables in this case are

$$\{x_n : n \in \mathbb{Z}\} \text{ satisfying } x_n x_{n-2} = \begin{cases} x_{n-1}^b + 1 \text{ if } n \text{ is odd} \\ x_{n-1}^c + 1 \text{ if } n \text{ is even} \end{cases}.$$

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If we let $x_1 = x_2 = 1$, we obtain $\{x_3, x_4, x_5, x_6\} = \{2, 5, 13, 34\}$.

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Laurent Phenomenon and Positivity

Theorem (Fomin-Zelevinsky 2002) For any cluster algebra, all cluster variables, i.e. the x_n's are Laurent polynomials (i.e. denominator is a monomial) in the initial cluster $\{x_1, x_2\}$.

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Example (b=c=3):

$$x_n x_{n-2} = x_{n-1}^r + 1.$$

If we let $x_1 = x_2 = 1$ and $r = 3$, then $\{x_n\}|_{n\geq 1} = 1, \quad 1, \quad 2, \quad 9, \quad 365,$ 5403014, 432130991537958813, 14935169284101525874491673463268414536523593057, ...

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Lee-Schiffler provided a combinatorial interpretaion for x_n's for any $r \geq 3$ in terms of colored subpaths of Dyck paths.

Lee-Li-Zelevinsky obtained more general combinatorial interpretation (for any $\mathcal{A}(b, c)$ and more than cluster variables) in terms of compatible pairs.

Finite versus Infinite Type

Let $\mathcal{A}(b,c)$ be the subalgebra of $\mathbb{Q}(x_1,x_2)$ generated by $\{x_n : n \in \mathbb{Z}\}$ with x_n's as above.

Remark: $\mathcal{A}(1,1)$, $\mathcal{A}(1,2)$, and $\mathcal{A}(1,3)$ are of finite type, i.e. $\{x_n\}$ is a finite set. However, for $bc \geq 4$, we get all x_n's are distinct.

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Example: For $\mathcal{A}(1,1)$, the cluster variables are

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Motivated by Positivity

Goal: Construct a vector-space basis for $\mathcal{A}(b, c)$, i.e. $\{\beta_i : i \in \mathcal{I}\}$ s.t.

(a) $\beta_i \cdot \beta_j = \sum_k c_k \beta_k$ with $c_k \geq 0$.

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(b) Cluster monomials appear as basis elements.

(i.e. $\beta = x_k^d x_{k+1}^e$ for $k \in \mathbb{Z}$ and $d, e \geq 0$)

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(b) Cluster monomials appear as basis elements.

(i.e. $\beta = x_k^d x_{k+1}^e$ for $k \in \mathbb{Z}$ and $d, e \geq 0$)

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Example: $\mathcal{A}(2,2)$, i.e. Affine Type, of Type $\widetilde{A}_1$

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If we let $x_1 = x_2 = 1$, then $\{x_3, x_4, x_5, x_6, x_7, \dots\} = \{2, 5, 13, 34, 89, \dots\}$.

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where $z_1 := z$, $z_2 := z^2 - 2$, and $z_k := z \cdot z_{k-1} - z_{k-2}$ for $k \geq 3$ satsifying the desired properties.

Goal: Construct a vector-space basis for $\mathcal{A}(b, c)$, i.e. $\{\beta_i : i \in \mathcal{I}\}$ s.t.

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Greedy Basis Defined

Definition Laurent series x is said to be **pointed** at $(a_1, a_2) \in \mathbb{Z}^2$ if

$$x = \frac{1}{x_1^{a_1} x_2^{a_2}} \sum_{p,q \geq 0} c(p,q) x_1^{bp} x_2^{cq}$$

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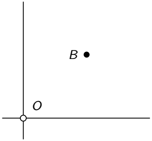
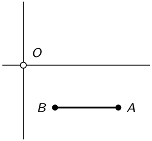
$$c(p,q) = \max\left(\sum_{k=1}^{p} (-1)^{k-1} c(p-k,q) \binom{a_2 - cq + k - 1}{k}, \sum_{k=1}^{q} (-1)^{k-1} c(p,q-k) \binom{a_1 - bp + k - 1}{k} \right).$$

This yields a unique element we denote as $\mathbf{x[a_1, a_2]}$.

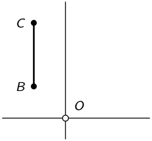
Pointed and Greedy Elements and their Terms Illustrated



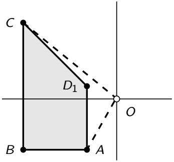
(1) $a_1, a_2 \leq 0$



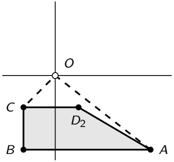
(2) $a_1 \leq 0 < a_2$



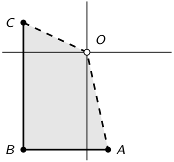
(3) $a_2 \leq 0 < a_1$



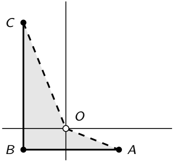
(4) $0 < ba_2 \leq a_1$



(5) $0 < ca_1 \leq a_2$



(6) $0 < a_1 < ba_2$,
$0 < a_2 < ca_1$,
(a_1, a_2) : non-imaginary root



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In particular, $x[-d, -e] = x_1^d x_2^e$ if $d, e \geq 0$ and other cluster monomials correspond to lattice points outside the third quadrant.
(8) For $a_1, a_2 \geq 0$, there are combiantorial formulas for $x[a_1, a_2]$ as

$$x[a_1, a_2] = \frac{1}{x_1^{a_1} x_2^{a_2}} \sum_{(S_1,S_2) \text{ compatible pairs in } D^{a_1 \times a_2}} x_1^{b|S_2|} x_2^{c|S_1|}.$$

In particular, $x[a_1, a_2]$ can be written as a Laurent polynomial with positive integer coefficients in any cluster.

Example: $\mathcal{A}(2,2)$, i.e. Affine Type, of Type $\widetilde{A}_1$ Revisited

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$z_k = x[k,k]$ and all other lattice points correspond to cluster monomials.

And Now For Something Completely Different

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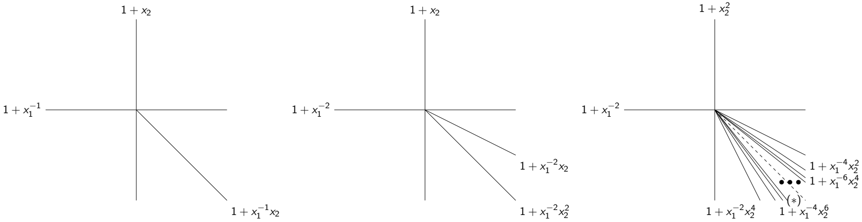
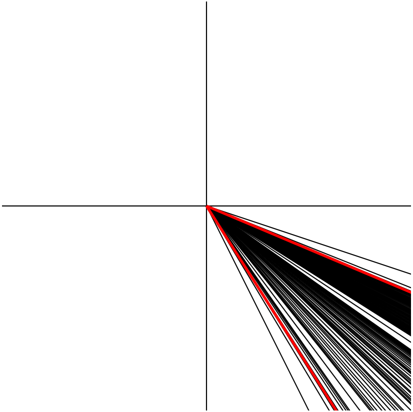


$$(*) = \frac{1}{(1-x_1^{-2}x_2^2)^4}.$$

And Now For Something Completely Different

A Scattering Diagram $\mathcal{D}(b,c)$ when $bc \geq 5$:

The Badlands! A cone where every wall of rational slope appears.



Definition Given a choice of $\mathcal{A}(b, c)$, a constructed scattering diagram $\mathcal{D}(b, c)$, a point $q \notin$ any wall, and an exponent $\overrightarrow{m} = (m_1, m_2)$, we construct $\vartheta_{q,\overrightarrow{m}}$ as follows:

1) Start with a line of initial slope m_2/m_1. Any parallel translate will do. We don't assume this fraction is written in reduced form and instead consider for example the vector $2\overrightarrow{m}$ to have larger momentum than $\overrightarrow{m}$.

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More precisely, if $\overrightarrow{d} = (d_1, d_2)$ is a primitive normal vector to the wall in question, then let $U = \overrightarrow{m} \cdot \overrightarrow{d} = m_1 d_1 + m_2 d_2$. Then the line can bend by adding $\overrightarrow{0}$, $\overrightarrow{d}$, $2\overrightarrow{d}$, $\ldots$, or $U\overrightarrow{d}$ to $\overrightarrow{m}$.

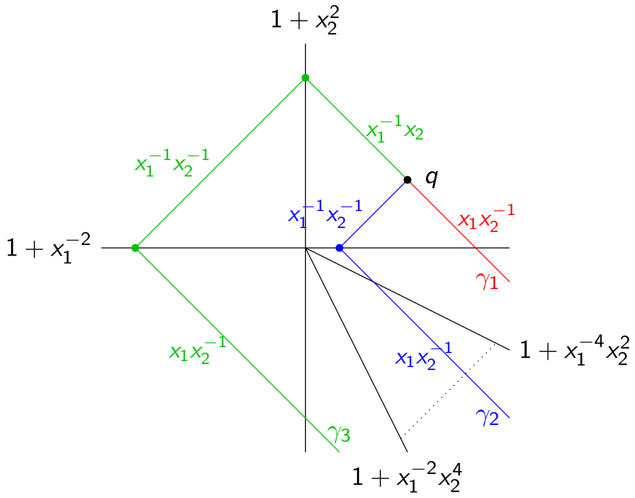
Broken Lines and Theta Functions

3) After choosing a sequence of such bends at each intersected wall, we consider the broken lines γ that end at the point q.

4) For each broken line γ, we obtain a Laurent monomial $x(\gamma) = x_1^{e_1} x_2^{e_2}$ where $\overrightarrow{e} = (e_1, e_2)$ is the final momentum/exponent of the broken line as it reaches q.

$$\vartheta_{q,\overrightarrow{m}} = \sum_{\text{all broken lines } \gamma \text{ starting with exponent } \overrightarrow{m} \text{ and ending at the point } q} x(\gamma)$$

Example for $\mathcal{A}(2,2)$



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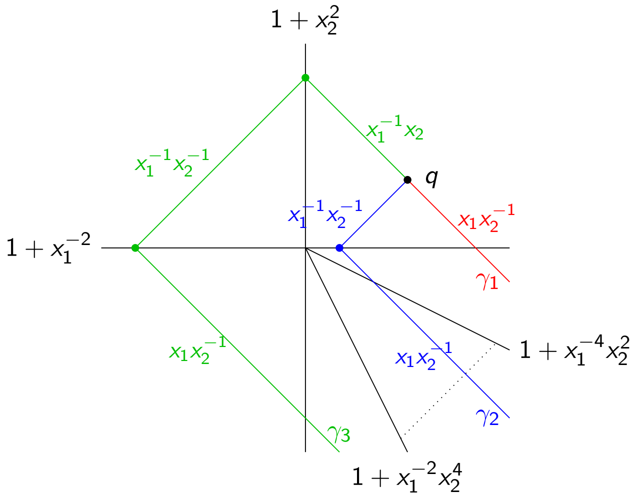


$$\vartheta_{q,\overrightarrow{(1,-1)}} = x_1x_2^{-1} + x_1^{-1}x_2^{-1} + x_1^{-1}x_2 = \frac{x_1^2+x_2^2+1}{x_1x_2} = z \in \mathcal{A}(2,2)$$

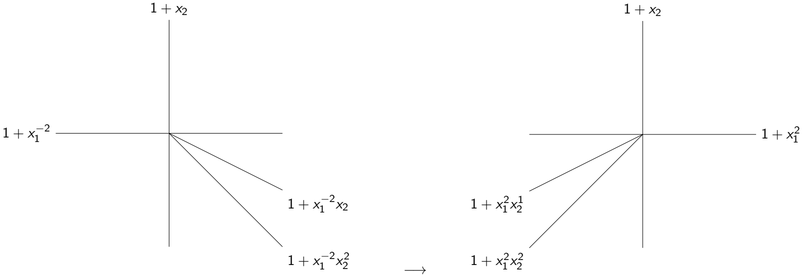
Theorem (Cheung-Gross-Muller-M-Rupel-Stella-Williams)

The greedy basis element $x[a_1, a_2] \in \mathcal{A}(b, c)$ equals the theta function (basis element) $\vartheta_{q, \overrightarrow{T(-a_1, -a_2)}}$ when

i) q is in the first quadrant generically

ii) drawing broken lines using the scattering diagram $\mathcal{D}(b, c)$

iii) T is a piece-wise linear transformation that simply pushes the **d**-vector fan into the **g**-vector fan by "pushing rays clockwise".



Sketch of Proof

Suppose a broken line γ passes through the point (q_1, q_2) with exponent/momentum $\overrightarrow{m} = (m_1, m_2)$ as it does.
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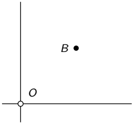
Consequently, we get on bounds on what the exponent of the Laurent monomial $x(\gamma) = x_1^{e_1} x_2^{e_2}$ can be.

Sketch of Proof

Allows us to build a Newton polygon

(more precisely a half-open quadrilateral)

of the support of $\vartheta_{q,\overrightarrow{T(-a_1,-a_2)}}$.



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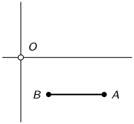

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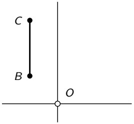

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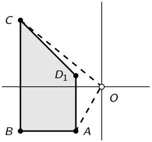

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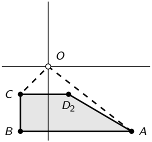

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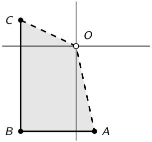
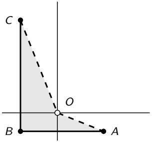


(6) $0 < a_1 < ba_2$,
$0 < a_2 < ca_1$,
(a_1, a_2) : non-imaginary root

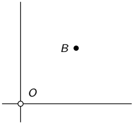


(6) $0 < a_1 < ba_2$,
$0 < a_2 < ca_1$,
(a_1, a_2) : imaginary root

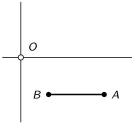
Sketch of Proof

$O = (0,0)$, $A = (-a_1 + ba_2, -a_2)$, $B = (-a_1, -a_2)$, $C = (-a_1, -a_2 + ca_1)$,
$D_1 = (-a_1 + ba_2, ca_1 - (bc+1)a_2)$, $D_2 = (ba_2 - (bc+1)a_1, -a_2 + ca_1)$.

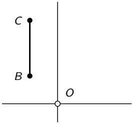
Then the support region of $\vartheta_{q,\overrightarrow{T(-a_1,-a_2)}}$ is



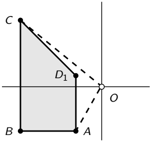
(1) $a_1, a_2 \le 0$



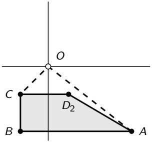
(2) $a_1 \le 0 < a_2$



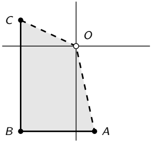
(3) $a_2 \le 0 < a_1$



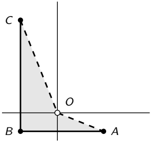
(4) $0 < ba_2 \le a_1$



(5) $0 < ca_1 \le a_2$



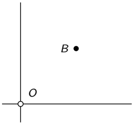
(6) $0 < a_1 < ba_2$,
$0 < a_2 < ca_1$,
(a_1, a_2) : non-imaginary root



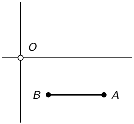
(6) $0 < a_1 < ba_2$,
$0 < a_2 < ca_1$,
(a_1, a_2) : imaginary root

Sketch of Proof

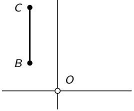
Scholium (based on Lee-Li-Zelevinsky's proof) Any Laurent polynomial with a support contained in one of these half-open quadrilaterals and with coefficient 1 on $x_1^{-a_1}x_2^{-a_2}$ **must be the greedy basis element** $x[a_1, a_2]$.



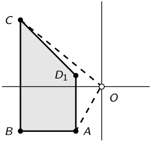
(1) $a_1, a_2 \leq 0$



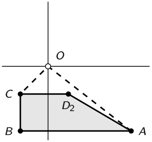
(2) $a_1 \leq 0 < a_2$



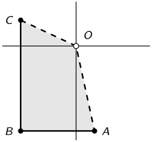
(3) $a_2 \leq 0 < a_1$



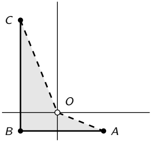
(4) $0 < ba_2 \leq a_1$



(5) $0 < ca_1 \leq a_2$



(6) $0 < a_1 < ba_2$, $0 < a_2 < ca_1$, (a_1, a_2) : non-imaginary root



(6) $0 < a_1 < ba_2$, $0 < a_2 < ca_1$, (a_1, a_2) : imaginary root

Thank You for Listening

Open Question 1: Find a combiantorial bijection between broken lines and compatible subsets.

Open Question 2: Reverse-engineer greedy bases in higher rank cluster algebras using theta functions.

Slides at `http://math.umn.edu/~musiker/Haiku.pdf`

"The Greedy Basis Equals the Theta Basis: A Rank Two Haiku"

(with Man Wai Cheung, Mark Gross, Greg Muller, Dylan Rupel, Salvatore Stella, and Harold Williams)

`http://arxiv.org/abs/1508.01404`