Convex incidences, neuroscience, and ideals

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(joint w/ R. Amzi Jeffs)
Combinatorial Ideals & Applications
AMS Spring Central Sectional Meeting

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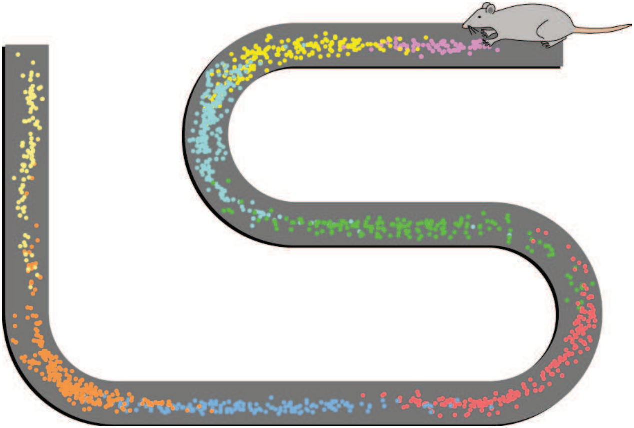
Biological Motivation

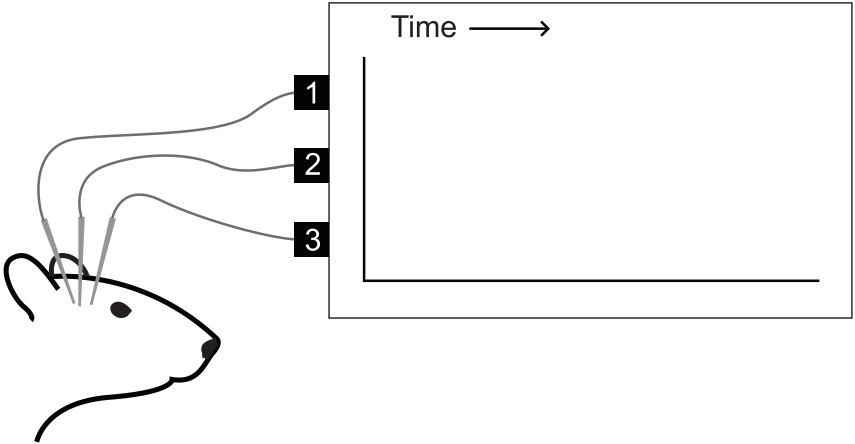
Place cells: Neurons which are active in a particular region of an animal's environment. (Nobel Prize 2014, Physiology or Medicine, O'Keefe/Moser-Moser)



https://upload.wikimedia.org/wikipedia/commons/5/5e/Place_Cell_Spiking_Activity_Example.png

Biological Motivation

How is data on place cells collected?



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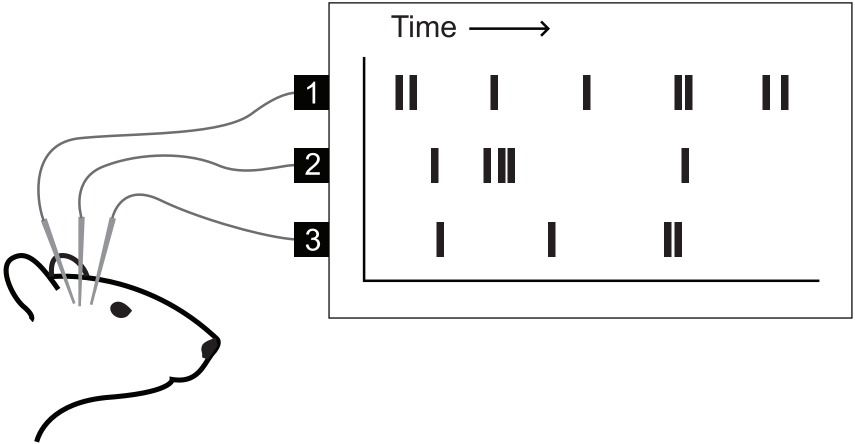

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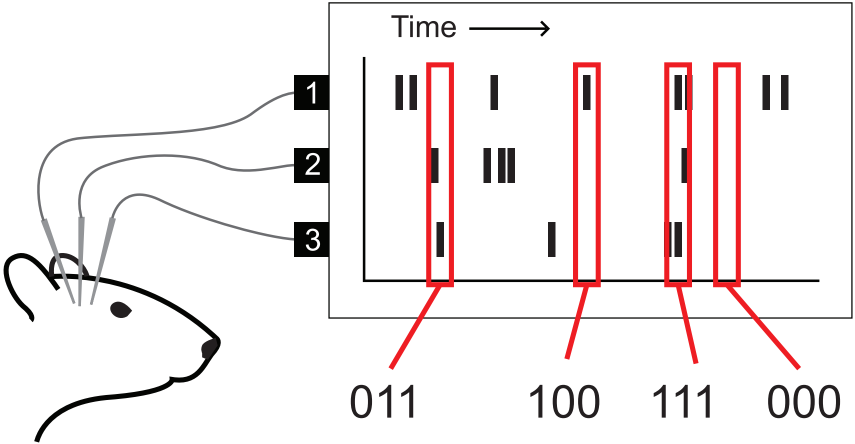

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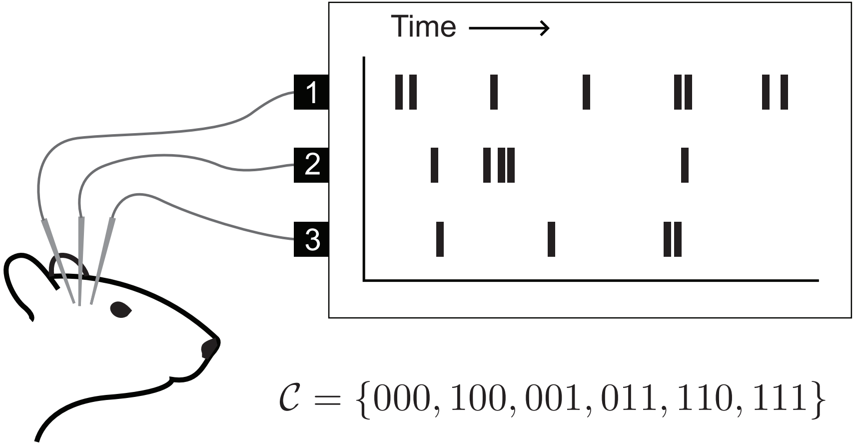
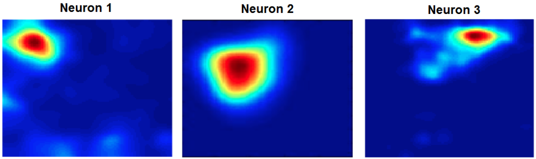


$$\mathcal{C} = \{000, 100, 001, 011, 110, 111\}$$

Mathematical Formulation

Neural codes capture an animal's response to a stimulus.

We assume that the receptive fields for place cells are open convex sets in Euclidean space.



Mathematical Formulation

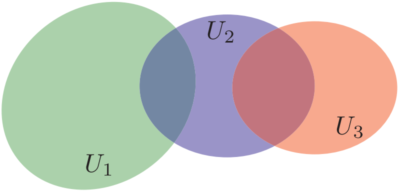
We associate collections of convex sets to binary codes.

Definition (Curto et. al, 2013)

Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a collection of convex open sets. The *code* of $\mathcal{U}$ is

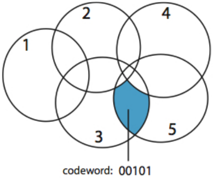
$$\mathcal{C}(\mathcal{U}) := \left\{ v \in \{0,1\}^n \,\middle|\, \bigcap_{v_i=1} U_i \setminus \bigcup_{v_j=0} U_j \neq \varnothing \right\}$$

$$\mathcal{U} = \{U_1, U_2, U_3\}$$



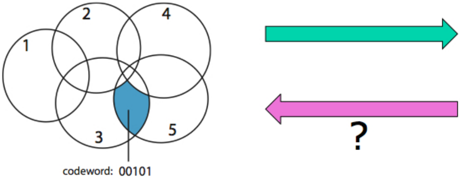
$$\mathcal{C}(\mathcal{U}) = \{000, 100, 010, 001, 110, 011\}$$

The Question



Code C:
00000 10000 01000 00100
00010 00001 11000 10100
01100 01010 00101 00011
11100 01110 01101 01011
00111 01111

The Question



Code C:

00000	10000	01000	00100
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01100	01010	00101	00011
11100	01110	01101	01011
00111	01111		

Definition

Let $\mathcal{C} \subseteq \{0,1\}^n$ be a code. If there exists a collection of convex open sets $\mathcal{U}$ so that $\mathcal{C} = \mathcal{C}(\mathcal{U})$ we say that $\mathcal{C}$ is *convex*. We call $\mathcal{U}$ a *convex realization* of $\mathcal{C}$.

Question

How can we detect whether a code $\mathcal{C}$ is convex?

Non-Example

Consider the code $\mathcal{C} = \{000, 100, 010, 110, 011, 101\}$

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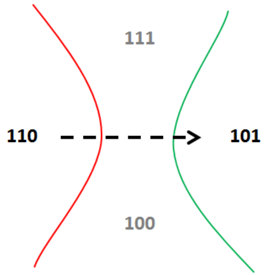

$\mathcal{C}$ is not realizable!

Classifying Convex Codes

Question

Can we find meaningful criteria that guarantee a code is convex?

Answer: Yes!

Classifying Convex Codes

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Answer: Yes!

- Simplicial complex codes (Curto et. al, 2013)
- Codes with $11\cdots1$ in them (Curto et. al, 2016)
- Intersection complete codes (Kronholm et. al, 2015)
- Many more (results from several papers)

Other Ideas....

Use Ideals!

An Algebraic Approach

We will work in the polynomial ring $\mathbb{F}_2[x_1, \ldots, x_n]$.

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Definition (CIVCY2013)

Let $v \in \{0,1\}^n$. The *indicator pseudomonomial* for v is

$$\rho_v := \prod_{v_i=1} x_i \prod_{v_j=0} (1 - x_j).$$

$\rho_{110} = x_1 x_2 (1 - x_3)$. Note that $\rho_v(u) = 1$ only if $u = v$.

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Definition (CIVCY2013)

Let $\mathcal{C} \subseteq \{0,1\}^n$ be a code. The *neural ideal* $J_{\mathcal{C}}$ of $\mathcal{C}$ is the ideal

$$J_{\mathcal{C}} := \langle \rho_v \mid v \notin \mathcal{C} \rangle.$$

Neural Ideal Example

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$$J_{\mathcal{C}} := \langle \rho_v \mid v \notin \mathcal{C} \rangle.$$

- $\mathcal{C} = \{000, 100, 010, 001, 011\}$
- $$\begin{aligned} J_{\mathcal{C}} = \langle \rho_v \mid v \notin \mathcal{C} \rangle &= \langle x_1x_2(1-x_3), x_1x_3(1-x_2), x_1x_2x_3 \rangle \\ &= \langle x_1x_2, x_1x_3(1-x_2) \rangle \end{aligned}$$

Canonical Form

Definition (CIVCY2013)

Let $J_{\mathcal{C}}$ be a neural ideal. The *canonical form* of $J_{\mathcal{C}}$ is the set of minimal pseudomonomials in $J_{\mathcal{C}}$ with respect to division. Equivalently :

$$CF(J_{\mathcal{C}}) := \{f \in J_{\mathcal{C}} \mid f \text{ is a PM and no proper divisor of } f \text{ is in } J_{\mathcal{C}}\}.$$

Canonical Form and Constructing Codes

Consider the code $C = \{00000, 10000, 01000, 00100, 00001, 11000, 10001, 01100, 00110, 00101, 00011, 11100, 00111\}$.

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$J_{\mathcal{C}} = \langle x_4(1-x_1)(1-x_2)(1-x_3)(1-x_5), x_1x_3(1-x_2)(1-x_4)(1-x_5), x_1x_4(1-x_2)(1-x_3)(1-x_5), x_2x_4(1-x_1)(1-x_3)(1-x_5), x_2x_5(1-x_1)(1-x_3)(1-x_4), x_1x_2x_4(1-x_3)(1-x_5), x_1x_2x_5(1-x_3)(1-x_4), x_1x_3x_4(1-x_2)(1-x_5), x_1x_3x_5(1-x_2)(1-x_4), x_1x_4x_5(1-x_2)(1-x_3), x_2x_3x_4(1-x_1)(1-x_5), x_2x_3x_5(1-x_1)(1-x_4), x_2x_4x_5(1-x_1)(1-x_3), x_2x_3x_4x_5(1-x_1), x_1x_3x_4x_5(1-x_2), x_1x_2x_4x_5(1-x_3), x_1x_2x_3x_5(1-x_4), x_1x_2x_3x_4(1-x_5), x_1x_2x_3x_4x_5\rangle$

Uggghhhh!

Canonical Form and Constructing Codes

Canonical Form (Minimal description!)

$$J_{\mathcal{C}} = \langle x_1x_3x_5, x_4(1-x_3)(1-x_5), x_1x_4, x_1x_3(1-x_2), x_2x_4, x_2x_5 \rangle$$

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- $x_1x_3x_5$

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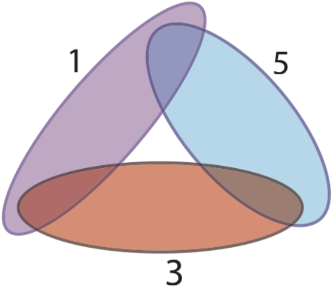
Canonical Form and Constructing Codes

The picture so far:



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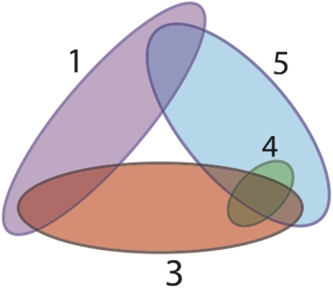

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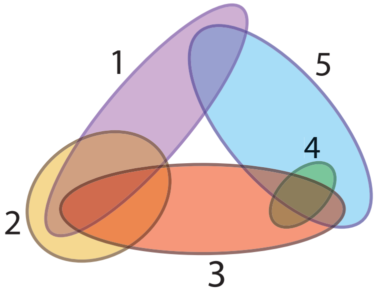
Canonical Form and Constructing Codes

Final picture:



The Neural Ideal in Summary

$$\mathcal{C} \longrightarrow J_{\mathcal{C}} \longrightarrow CF(J_{\mathcal{C}})$$

We associate codes to neural ideals, and use the canonical form to compactly present the neural ideal and encode information about the code and its realizations.

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We associate codes to neural ideals, and use the canonical form to compactly present the neural ideal and encode information about the code and its realizations.

We hope to understand convex codes by examining neural ideals and their canonical forms.

Homomorphisms Respecting Neural Ideals

R. Amzi Jeffs, '16

Definition

We say a homomorphism $\phi : \mathbb{F}_2[n] \to \mathbb{F}_2[m]$ *respects neural ideals* if for every $\mathcal{C} \subseteq \{0,1\}^n$ there exists $\mathcal{D} \subseteq \{0,1\}^n$ so that

$$\phi(J_{\mathcal{C}}) = J_{\mathcal{D}}.$$

That is, if ϕ maps neural ideals to neural ideals.

Can we classify all such homomorphisms? Do they have geometric meaning?

Homomorphisms Respecting Neural Ideals

Restriction: Mapping $x_i \mapsto 1$ or $x_i \mapsto 0$ for some i.

- $x_i \mapsto 1$ corresponds with replacing each U_j by $U_j \cap U_i$.
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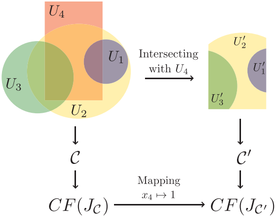

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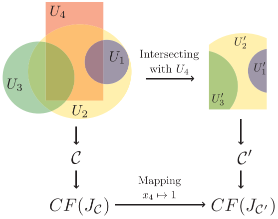

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- Corresponds to taking the complement of U_i.

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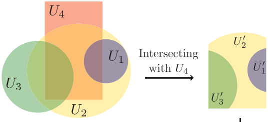


$$\begin{array}{ccc} \mathcal{C} & & \mathcal{C}' \\ \downarrow & & \downarrow \\ CF(J_{\mathcal{C}}) & \xrightarrow[\text{Mapping}]{x_4 \mapsto 1} & CF(J_{\mathcal{C}'}) \end{array}$$

Bit Flipping: Mapping $x_i \mapsto 1 - x_i$ for some i.

- Corresponds to taking the complement of U_i.

Permutation: Permuting labels on the variables in $\mathbb{F}_2[n]$.

- Corresponds to permuting labels on the sets in a realization.

Classifying Homomorphisms Respecting Neural Ideals

Theorem (Jeffs, O.)

Let $\phi : \mathbb{F}_2[n] \to \mathbb{F}_2[m]$ be a homomorphism respecting neural ideals. Then ϕ is the composition of the three types of maps previously described:

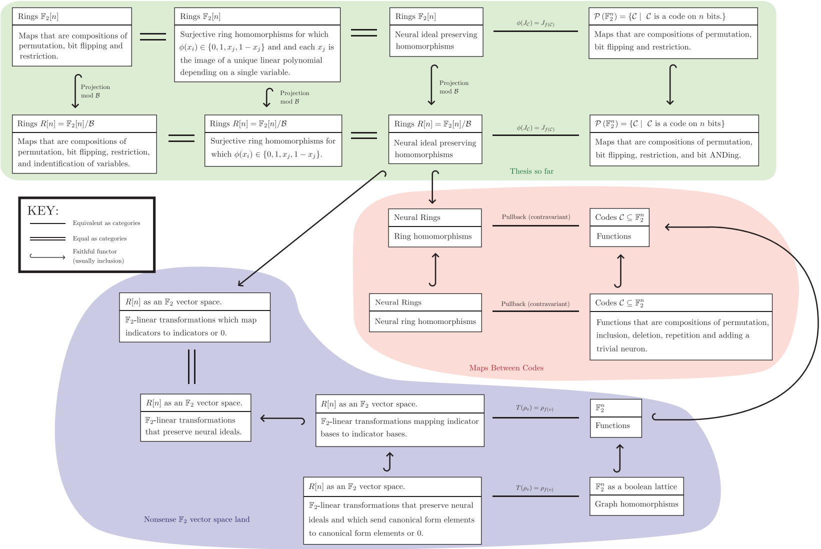
- *Permutation*
- *Restriction*
- *Bit flipping*

Moreover, there is an algorithm to present ϕ as such a composition.

Homomorphisms Respecting Neural Ideals: Proof Idea

1. If $\phi : \mathbb{F}_2[n] \to \mathbb{F}_2[m]$ respects neural ideals if and only if ϕ is
   - surjective, and
   - sends pseudonomials to pseudomonomials or 0
2. $\phi(x_i) \in \{x_j, 1 - x_j, 0, 1\}$, and for every $j \in [m]$ there is a unique $i \in [n]$ so that $\phi(x_i) \in \{x_j, 1 - x_j\}$.
3. (Carefully) piece things together variable by variable.

Mapping the Work



Conclusion

In This Talk:

- We associated polynomial ideals to codes.
- We used these ideals to understand codes and their realizations
- We described a class of homomorphisms which play nicely with these ideals. These homomorphisms can be used to understand convex codes, and also computationally construct them.

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What's Next?

- How do maps respecting neural ideals affect canonical forms?
- What other algebraic techniques can be leveraged?

Thank You!