Invariant ideals and the Delta conjecture

Brendon Rhoades
(joint with Jim Haglund, Jeff Remmel, and Andy Wilson)

UCSD

Outline

1. inv/maj-Equidistribution for permutations
2. inv/maj-Equidistribution for ordered set partitions
3. The Delta Conjecture
4. Open Problems: Invariant Ideals

Permutation Statistics

Given $\pi = \pi_1\pi_2\ldots\pi_n \in S_n$,

$$\mathrm{inv}(\pi) = \#\{i < j \ : \ \pi_i > \pi_j\},$$
$$\mathrm{maj}(\pi) = \sum_{\pi_i > \pi_{i+1}} i.$$

$$\mathrm{inv}(31452) = \#\{(1,3),(2,3),(2,4),(2,5)\} = 4$$
$$\mathrm{maj}(31452) = 1 + 4 = 5.$$

Thm: [MacMahon] The statistics inv and maj are equidistributed on S_n. In fact, their common distribution is

$$\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} = \sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} = [n]!_q$$
$$:= (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

Bijective Proofs

Foata and Carlitz gave *bijective* proofs of MacMahon's Theorem by finding bijections

$$\phi_F, \phi_C : S_n \to S_n$$

sending inv to maj.

We'll describe a 'cyclic variant' of the Carlitz bijection.

Obs: Let $c = (n, n-1, \ldots, 1) \in S_n$. If $\pi = \pi_1 \pi_2 \ldots \pi_n \in S_n$ and $\pi_n \neq 1$, then

$$\mathrm{maj}(c.\pi) = \mathrm{maj}(\pi) + 1.$$

$$254381796 \xrightarrow{c\cdot(-)} 143279685$$

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \to S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193		

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \to S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193	1	
72564813		

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \rightarrow S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193	1	
72564813	2	
7256413		

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \to S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193	1	
72564813	2	
7256413	6	
256413		

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \to S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193	1	
72564813	2	
7256413	6	
256413	3	
25413		

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \to S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193	1	
72564813	2	
7256413	6	
256413	3	
25413	3	
2413		

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \to S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193	1	
72564813	2	
7256413	6	
256413	3	
25413	3	
2413	2	
213		

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \to S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193	1	
72564813	2	
7256413	6	
256413	3	
25413	3	
2413	2	
213	0	
21		

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \to S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193	1	
72564813	2	
7256413	6	
256413	3	
25413	3	
2413	2	
213	0	
21	1	
1		

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \to S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193	1	
72564813	2	
7256413	6	
256413	3	
25413	3	
2413	2	
213	0	
21	1	
1		1

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \to S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193	1	
72564813	2	
7256413	6	
256413	3	
25413	3	
2413	2	
213	0	
21	1	$c^1.(12)$
1		1

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \to S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193	1	
72564813	2	
7256413	6	
256413	3	
25413	3	
2413	2	
213	0	
21	1	$c^1.(12) = 21$
1		1

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \to S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193	1	
72564813	2	
7256413	6	
256413	3	
25413	3	
2413	2	
213	0	$c^0.(213)$
21	1	$c^1.(12) = 21$
1		1

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \to S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193	1	
72564813	2	
7256413	6	
256413	3	
25413	3	
2413	2	
213	0	$c^0.(213) = 213$
21	1	$c^1.(12) = 21$
1		1

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \to S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193	1	
72564813	2	
7256413	6	
256413	3	
25413	3	
2413	2	$c^2.(2134)$
213	0	$c^0.(213) = 213$
21	1	$c^1.(12) = 21$
1		1

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \to S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193	1	
72564813	2	
7256413	6	
256413	3	
25413	3	
2413	2	$c^2.(2134) = 4312$
213	0	$c^0.(213) = 213$
21	1	$c^1.(12) = 21$
1		1

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \to S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193	1	
72564813	2	
7256413	6	
256413	3	
25413	3	$c^3.(43125) = 15342$
2413	2	$c^2.(2134) = 4312$
213	0	$c^0.(213) = 213$
21	1	$c^1.(12) = 21$
1		1

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \to S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193	1	
72564813	2	
7256413	6	
256413	3	$c^3.(153426) = 426153$
25413	3	$c^3.(43125) = 15342$
2413	2	$c^2.(2134) = 4312$
213	0	$c^0.(213) = 213$
21	1	$c^1.(12) = 21$
1		1

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \to S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193	1	
72564813	2	
7256413	6	$c^6.(4261537) = 5372641$
256413	3	$c^3.(153426) = 426153$
25413	3	$c^3.(43125) = 15342$
2413	2	$c^2.(2134) = 4312$
213	0	$c^0.(213) = 213$
21	1	$c^1.(12) = 21$
1		1

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \to S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193	1	
72564813	2	$c^2.(53726418) = 31584276$
7256413	6	$c^6.(4261537) = 5372641$
256413	3	$c^3.(153426) = 426153$
25413	3	$c^3.(43125) = 15342$
2413	2	$c^2.(2134) = 4312$
213	0	$c^0.(213) = 213$
21	1	$c^1.(12) = 21$
1		1

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \to S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193	1	$c^1.(315842769) = 294731658$
72564813	2	$c^2.(53726418) = 31584276$
7256413	6	$c^6.(4261537) = 5372641$
256413	3	$c^3.(153426) = 426153$
25413	3	$c^3.(43125) = 15342$
2413	2	$c^2.(2134) = 4312$
213	0	$c^0.(213) = 213$
21	1	$c^1.(12) = 21$
1		1

Cyclic Carlitz Bijection

Can build a bijection $\phi : S_n \to S_n$ such that $\mathrm{inv}(\pi) = \mathrm{maj}(\phi(\pi))$.

π	inv lost/ maj gained	$\phi(\pi)$
725648193	1	$c^1.(315842769) = 294731658$
72564813	2	$c^2.(53726418) = 31584276$
7256413	6	$c^6.(4261537) = 5372641$
256413	3	$c^3.(153426) = 426153$
25413	3	$c^3.(43125) = 15342$
2413	2	$c^2.(2134) = 4312$
213	0	$c^0.(213) = 213$
21	1	$c^1.(12) = 21$
1		1

Q: Any modern applications of this idea?

Ordered Set Partitions

Def: An *ordered set partition* of size n is a set partition of $[n]$ with a total order on its blocks.

Def: The *shape* of $\sigma = (B_1|B_2|\ldots|B_k) \models [n]$ is $(|B_1|, |B_2|, \ldots, |B_k|) \models n$.

$$\begin{aligned}\sigma &= \{2,3,7\} \prec \{1\} \prec \{4,6\} \prec \{5\} \\ &= (237|1|46|5) \models [7]\end{aligned}$$

$$\mathrm{shape}(\sigma) = (3,1,2,1) \models 7.$$

Statistics for Ordered Set Partitions

Def: [Steingrímsson] An *inversion* in $\sigma = (B_1|B_2|\dots|B_k) \models [n]$ is a pair $i < j$ such that $j \in B_k$ and $i = \min(B_\ell)$ for $k < \ell$.

$$\mathrm{inv}(237|1|46|5) = \#\{(1,2),(1,3),(1,7),(4,7),(5,7),(5,6)\} = 6$$

Def: [Loehr, Haglund-Remmel-Wilson] For $\sigma \models [n]$, $\mathrm{minimaj}(\sigma)$ is

$$\min\{\mathrm{maj}(\pi) : \pi \in S_n \text{ is obtained by rearranging within blocks of } \sigma\}$$

$$\begin{aligned}\mathrm{minimaj}(15|2|34) &= \min\{\mathrm{maj}(15234), \mathrm{maj}(51234), \mathrm{maj}(15243), \mathrm{maj}(51243)\} \\ &= \min\{2,1,6,5\} = 1.\end{aligned}$$

Equidistribution for Ordered Set Partitions

For $\alpha \models n$,

$$\mathcal{OP}_\alpha := \{\sigma \models [n] \,:\, \mathrm{shape}(\sigma) = \alpha\}.$$

Thm: [HRRW] Let $\alpha = (\alpha_1, \ldots, \alpha_k) \models n$. The statistics inv and $\mathrm{minimaj}$ are equidistributed on $\mathcal{OP}_\alpha$.

$$\sum_{\sigma \in \mathcal{OP}_\alpha} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in \mathcal{OP}_\alpha} q^{\mathrm{minimaj}(\sigma)} = F_\alpha(q),$$

where

$$F_\alpha(q) = \prod_{i=1}^{k} \left[\binom{\alpha_i - 1}{\alpha_i - 1} + \binom{\alpha_i}{\alpha_i - 1} q + \cdots + \binom{\alpha_1 + \cdots + \alpha_i - 1}{\alpha_i - 1} q^{\alpha_1 + \cdots + \alpha_{i-1}} \right]$$

- We have $F_\alpha(1) = \binom{n}{\alpha_1, \ldots, \alpha_k}$, but $F_\alpha(q)$ is *not* the standard q-multinomial $\begin{bmatrix} n \\ \alpha_1, \ldots, \alpha_k \end{bmatrix}_q = \frac{[n]!_q}{[\alpha_1]!_q \cdots [\alpha_k]!_q}$.

Ordered Multiset Partitions

Def: An *ordered multiset partition* is a sequence of sets $\mu = (M_1|M_2|\ldots|M_k)$.

Ex: $\mu = (136|14|236|6)$ has shape $(3, 2, 3, 1)$ and *weight* $(2, 1, 2, 1, 0, 3)$.

inv and $\mathrm{minimaj}$ still make sense for ordered multiset partitions.

Q: Are they equidistributed?

Equidistribution for Multiset Partitions

For compositions α, β, let

$\mathcal{OP}_{\alpha}^{\beta} = \{$all ordered multiset partitions of shape α and weight $\beta\}$.

Guess: The statistics inv and $\mathrm{minimaj}$ are equidistributed on $\mathcal{OP}_{\alpha}^{\beta}$.

Equidistribution for Multiset Partitions

For compositions α, β, let

$\mathcal{OP}_{\alpha}^{\beta} = \{$all ordered multiset partitions of shape α and weight $\beta\}$.

Guess: The statistics inv and $\mathrm{minimaj}$ are equidistributed on $\mathcal{OP}_{\alpha}^{\beta}$.

Nope. For $\alpha = (2, 1, 2)$ and $\beta = (2, 2, 1)$,

$$\mathrm{inv} \text{ distribution} = q + 2q^2 + q^3 + q^4$$
$$\mathrm{minimaj} \text{ distribution} = q + q^2 + 2q^3 + q^4.$$

Salvaged Equidistribution for Multiset Partitions

For a composition β and $k > 0$, let

$\mathcal{OP}_k^\beta := \{$all ordered multiset partitions with k blocks of weight $\beta\}$.

Thm: [HRRW] The statistics inv and $\mathrm{minimaj}$ *are* equidistributed on $\mathcal{OP}_k^\beta$.

Q: Applications?

The Diagonal Coinvariants

Let S_n act on $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ *diagonally*:

$$\pi.x_i := x_{\pi(i)}$$
$$\pi.y_i := y_{\pi(i)}.$$

Def: The *diagonal coinvariant module* is the bigraded S_n-representation

$$DR_n := \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]/\langle \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]_+^{S_n} \rangle.$$

Thm: [Haiman] We have $\dim(DR_n) = (n+1)^{n-1}$. In fact (up to sign twist), DR_n is isomorphic to the permutation action of S_n on size n parking functions.

The Former Shuffle Conjecture

Q: What is the bigraded S_n-module structure of DR_n?

Thm: [Haiman] The bigraded Frobenius series of DR_n is (up to a twist) $\nabla(e_n)$, where ∇ is the Bergeron-Garsia nabla operator on symmetric functions (a *Macdonald eigenoperator*).

Thm: [Carlsson-Mellit] ('Shuffle Conjecture') We have that

$$\nabla(e_n) = \sum_{P \in \mathrm{Park}_n} q^{\mathrm{area}(P)} t^{\mathrm{dinv}(P)} F_{\mathrm{iDes}(P)}.$$

Our equidistribution results give evidence for a generalization of the Shuffle Conjecture.

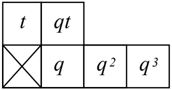
Delta Operators

- Λ_n = symmetric functions in $(x_1, x_2, \dots)$ of degree n.
- $\{\widetilde{H}_\mu : \mu \vdash n\}$ = modified Macdonald basis.
- $f = f(x_1, x_2, \dots)$ a symmetric function.

Def: $\Delta'_f : \Lambda_n \to \Lambda_n$ is the Macdonald eigenoperator defined by

$$\Delta'_f : \widetilde{H}_\mu \mapsto f(\dots, q^i t^j, \dots)\widetilde{H}_\mu,$$

where (i, j) range over all cells $\neq (0,0)$ of the Ferrers diagram of μ.
Ex: $\mu = (4, 2) \vdash 6$.



$$\Delta'_f(\widetilde{H}_\mu) = f(q, q^2, q^3, t, qt)\widetilde{H}_\mu$$

Fact: $\Delta'_{e_{n-1}} = \nabla$.

Delta Conjecture

Conj: [Haglund, Remmel, Wilson] For any $n > k \geq 0$ we have

$$\Delta'_{e_k}(e_n) = \{z^{n-k-1}\}\left[\sum_{P \in \mathcal{LD}_n} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} \prod_{i: a_i(P) > a_{i-1}(P)} \left(1 + z/t^{a_i(P)}\right) x^P\right]$$

$$= \{z^{n-k-1}\}\left[\sum_{P \in \mathcal{LD}_n} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} \prod_{i \in \mathrm{Val}(P)} \left(1 + z/t^{d_i(P)+1}\right) x^P\right],$$

where $\{z^{n-k-1}\}$ extracts the coefficient of z^{n-k-1}.

Rmk: When $k = n - 1$, this is the Shuffle Conjecture.

Def: Let $\mathrm{Val}_{n,k}(x; q, t)$ denote the expression in the second line.

Equidistribution and the Delta Conjecture

Conj: For all $n > k \geq 0$,

$$\Delta'_{e_k}(e_n) = \mathrm{Val}_{n,k}(x; q, t).$$

Our results on ordered multiset partitions imply that ...

- We have $\mathrm{Val}_{n,k}(x; q, 0) = \mathrm{Val}_{n,k}(x; 0, q)$.
- The quasisymmetric functions $\mathrm{Val}_{n,k}(x; q, 0)$ and $\mathrm{Val}_{n,k}(x; 0, q)$ are symmetric.

Invariant Ideals

S_n acts on $\mathbb{C}[x_1, \ldots, x_n]$ by coordinate permutation.

S_n also acts on $\mathbb{C}[x_1, \ldots, x_n]/\langle \mathbb{C}[x_1, \ldots, x_n]_+^{S_n} \rangle$.

Thm: (Lusztig, Stanley) Let $\lambda \vdash n$. The λ-isotypic component of $\mathbb{C}[x_1, \ldots, x_n]/\langle \mathbb{C}[x_1, \ldots, x_n]_+^{S_n} \rangle$ has Hilbert polynomial f^λ times

$$\sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \mathrm{Val}_{n,n-1}(x; 0, q).$$

Prop: [HRRW] The coefficient of s_λ in $\mathrm{Val}_{n,k}(x; 0, q)$ is

$$\sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T) + \binom{n-k}{2} - (n-k)\mathrm{des}(T)} \begin{bmatrix} \mathrm{des}(T) \\ n-k \end{bmatrix}_q.$$

Open: Find a natural graded module whose Frobenius series is $\mathrm{Val}_{n,k}(x; 0, q)$.

Thanks for listening!!