Matrix Completion, Free Resolutions, and Sums of Squares

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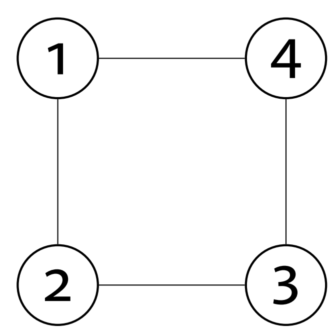

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Matrix Completion

Throughout, we fix a simple graph $G = ([n], E)$.

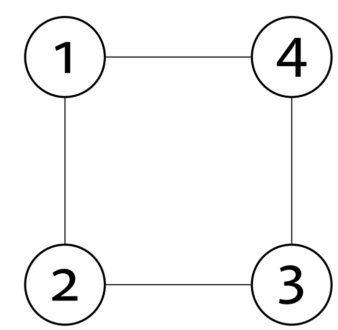
This encodes the coordinate projection

$$\pi_G : \begin{cases} \mathbb{S}^n & \to \quad \mathbb{R}^n \oplus \mathbb{R}^{\#E} \\ (a_{ij}) & \mapsto \quad (a_{11}, \ldots, a_{nn}) \oplus (a_{ij} : \{i, j\} \in E) \end{cases}$$

$\mathbb{S}^n$: real vector space of symmetric $n \times n$ matrices.

Example. *Let G be the four cycle.*

$$\pi_G \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & * & a_{14} \\ a_{12} & a_{22} & a_{23} & * \\ * & a_{23} & a_{33} & a_{34} \\ a_{14} & * & a_{34} & a_{44} \end{pmatrix}$$



NB: Think of $\pi_G(a_{ij})$ as a partial matrix.

Goal: Complete a G-partial matrix to a **positive semidefinite** symmetric matrix (and control the rank of the completion).

Geometry of positive semidefinite matrix completion

Goal: Describe the image of the cone $\mathbb{S}^n_{\geq 0}$ of positive semidefinite quadratic forms under the projection π_G.
$\mathbb{S}^n_{\geq 0}$: convex cone of quadratic forms $\sum_{i,j} a_{ij} x_i x_j$ such that $\sum_{i,j} a_{ij} p_i p_j \geq 0$ for all $(p_1, \ldots, p_n) \in \mathbb{R}^n$.

Theorem (Diagonalization of Quadratic Forms). *A quadratic form $q \in \mathbb{R}[x_1, \ldots, x_n]$ is positive semidefinite if and only if it is a sum of squares of linear forms after a change of basis.*

Question: Is $\pi_G(\mathbb{S}^n_{\geq 0})$ a cone of sums of squares?

Stanley-Reisner Ideals

Recall $G = ([n], E)$ is fixed.
Let $I_G = \langle x_i x_j \colon \{i, j\} \notin E \rangle \subset \mathbb{R}[x_1, \ldots, x_n]$

Then I_G is the Stanley-Reisner ideal of the **clique complex** of the graph:
simplicial complex Δ on $[n]$, where $H \in \Delta$ if and only if the induced subgraph of G on the vertices in H is complete.
So a monomial $x_H = \prod_{i \in H} x_i$ is in I_G if and only if $G|_H$ is not a complete graph, i.e. there are $i, j \in H$ such that $\{i, j\} \notin E$. But then $x_i x_j \in I_G$ and $x_i x_j | x_H$.

Note: The degree 2 part of I_G is the kernel of π_G.
The quadratic form $2x_i x_j$ corresponds to the symmetric matrix $E_{ij} + E_{ji}$.

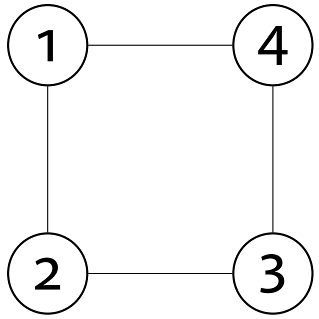
Subspace Arrangements

Set $X_G = \mathcal{V}(I_G) \subset \mathbb{P}^{n-1}$, the subscheme associated with the Stanley-Reisner ideal I_G.

$$X_G = \bigcup_{K \subset G} \operatorname{span}\{e_i : i \in K\},$$

where K runs over all complete subgraphs of G.

Example. *Let G be the four cycle.*
Then $I_G = \langle x_1x_3, x_2x_4\rangle$ and X_G is the union of four lines in $\mathbb{P}^3$.



Observation: The projection π_G is the degree 2 part of the map of coordinate rings $\mathbb{R}[x_1, \ldots, x_n] \to \mathbb{R}[X_G]$.

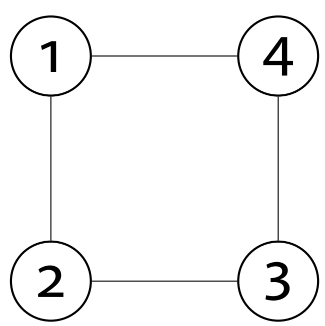
Proposition. *The image $\pi_G(\mathbb{S}^n_{\geq 0})$ of the cone of positive semidefinite quadratic forms is the cone Σ_{X_G} of all quadratic forms in $\mathbb{R}[X_G]_2$ that are sums of squares of linear forms.*

Summary of the setup

Let $G = ([n], E)$ be a simple graph and let $I_G = \langle x_i x_j \colon \{i, j\} \notin E\rangle$ be the Stanley-Reisner ideal of the clique complex of G.

Then the coordinate projection $\pi_G \colon \mathbb{S}^n \to \mathbb{R}^n \oplus \mathbb{R}^{\#E}$ is equal to the degree 2 part of the map $\mathbb{R}[x_1, \ldots, x_n] \to \mathbb{R}[X_G] = \mathbb{R}[x_1, \ldots, x_n]/I_G$ of homogeneous coordinate rings.

Therefore, the image $\pi_G(\mathbb{S}^n_{\geq 0})$ is the cone of sums of squares $\Sigma_{X_G} \subset \mathbb{R}[X_G]_2$ on X_G.



$$\begin{pmatrix} a_{11} & a_{12} & * & a_{14} \\ a_{12} & a_{22} & a_{23} & * \\ * & a_{23} & a_{33} & a_{34} \\ a_{14} & * & a_{34} & a_{44} \end{pmatrix}$$

$$I_G = \langle x_1 x_3, x_2 x_4 \rangle$$

Sums of Squares and Nonnegative Polynomials

Observation: Every sum of squares of linear forms is a nonnegative quadric.

$$q(p) = \sum_{i=1}^{r} \ell_i^2(p) = \sum_{i=1}^{r} (\ell_i(p))^2$$

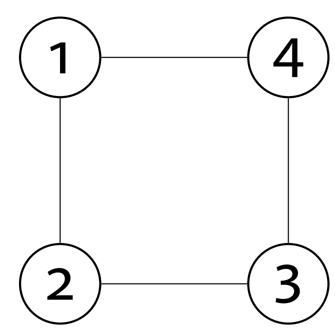
We write P_{X_G} for the cone of nonnegative quadrics.

Proposition (Obvious necessary condition). *Assume $\pi_G(A) \in \Sigma_{X_G}$. Then every completely specified symmetric submatrix of A is positive semidefinite.*

Proof. $\Sigma_{X_G} \subset P_{X_G}$. ∎

Example. *Let G be the four cycle. Then*

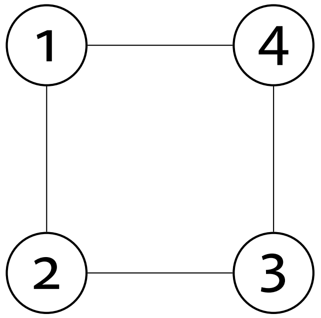
$$\pi_G(a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & * & a_{14} \\ a_{12} & a_{22} & a_{23} & * \\ * & a_{23} & a_{33} & a_{34} \\ a_{14} & * & a_{34} & a_{44} \end{pmatrix}.$$



The four fully specified submatrices correspond to the four maximal cliques of G, namely the edges.

Question: When is this obvious necessary condition sufficient?

Two long lost friends



Recall: $G = ([n], E)$ is a simple graph.
A graph G is **chordal** if every cycle in G of length at least 4 has a chord.

Theorem (Blekherman-Sinn-Velasco). *Every nonnegative quadric is a sum of squares on X_G (in symbols $\Sigma_{X_G} = P_{X_G}$) if and only if I_G is 2-regular.*

2**-regular**: The minimal free resolution is linear:

$$\cdots \xrightarrow{\varphi_{t+1}} F_t \xrightarrow{\varphi_t} F_{t-1} \xrightarrow{\varphi_{t-1}} \cdots \xrightarrow{\varphi_1} F_0 \to I \to 0$$

Theorem (Fröberg). *The monomial ideal I_G is 2-regular if and only if G is chordal.*

Theorem (Agler-Helton-McCullough-Rodman) and (Paulsen-Power-Smith). *The obvious necessary condition for positive semidefinite matrix completion is sufficient (equivalently, $\Sigma_{X_G} = P_{X_G}$) if and only if G is chordal.*

Minimal Free Resolutions and Matrix Completion

The dual convex cone to Σ_{X_G} is

$$\Sigma_{X_G}^{\vee} = \{\ell \in \mathbb{R}[X_G]_2^* : \ell(f) \geq 0 \text{ for all } f \in \Sigma_{X_G}\}.$$

Convex Biduality: If $\Sigma_{X_G} \subsetneq P_{X_G}$, then dually $P_{X_G}^{\vee} \subsetneq \Sigma_{X_G}^{\vee}$. So there is an extreme ray $\ell \in \Sigma_{X_G}$ which is not in P_{X_G}.

We can identify $\mathbb{R}[X_G]_2^* = (\mathbb{R}[x_1, \ldots, x_n]_2 / I_G)^*$ with a space of symmetric matrices: $I_G^{\perp} \subset \mathbb{R}[x_1, \ldots, x_n]_2^*$.

What is the smallest rank of an extreme ray $\ell \in \Sigma_{X_G} \setminus P_{X_G}$?

Theorem (Blekherman-Sinn-Velasco). *Suppose m is the Green-Lazarsfeld index of I_G, i.e. the largest integer p such that I_G has property $N_{2,p}$ (the minimal free resolution is linear for p steps). The smallest rank of an extreme ray $\ell \in \Sigma_{X_G} \setminus P_{X_G}$ is $m+1$. In terms of the graph, it is the smallest number $m \geq 1$ such that G contains an induced cycle of length $m+3$.*

Theorem (Eisenbud-Green-Hulek-Popescu). *The Green-Lazarsfeld index of I_G is the largest p such that G has no induced cycle of length at most $p+2$.*

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Thank you