Combinatorial dynamics of monomial ideals

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Abstract

We introduce the notion of combinatorial dynamics on algebraic ideals by translating combinatorial results involving the rowmotion action on order ideals of posets to the setting of monomial ideals.

Rowmotion

Given a finite poset P, the *rowmotion* of an order ideal $I \in J(P)$ is defined as the order ideal generated by the minimal elements of P not in I. Partially ordering the monomials of $R = K[x_1, \ldots, x_n]$ by divisibility, we can thus define rowmotion for one algebraic ideal with respect to another.

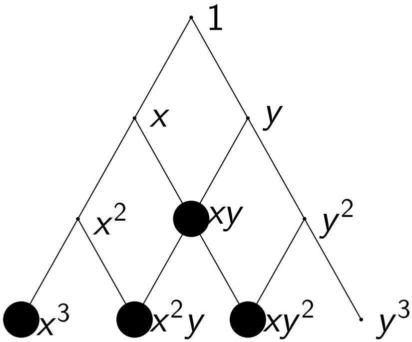
Rowmotion

Definition

Let I and J be monomial ideals of $R = K[x_1, \ldots, x_n]$. If $I \supset J$, then the *(ideal) rowmotion of I with respect to J* is the ideal of R generated by the maximal (with respect to divisibility) monomials in R not in I, together with the generators of J.

In our theorems, our base algebraic ideal J will be artinian, so that the set of standard monomials (monomials not in J) is finite; this corresponds to the case of finite posets. But artinian need not be an assumption in the definition.

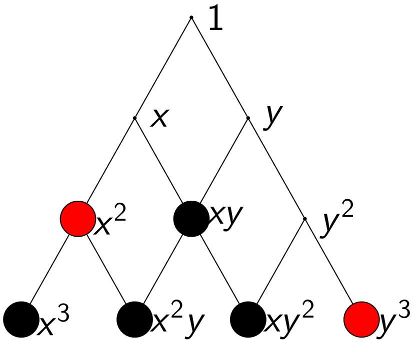
Rowmotion example



$$I = \langle x^3, xy, y^4 \rangle$$

$$J = \langle x^4, x^3y, x^2y^2, xy^3, y^4 \rangle$$

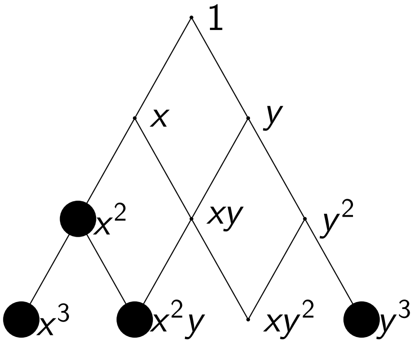
Rowmotion example



$$\mathrm{Row}(I) = \langle x^2, y^3 \rangle$$

$$J = \langle x^4, x^3y, x^2y^2, xy^3, y^4 \rangle$$

Rowmotion example



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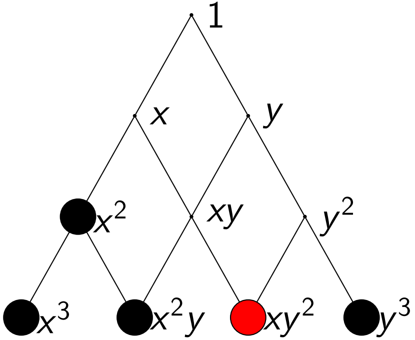
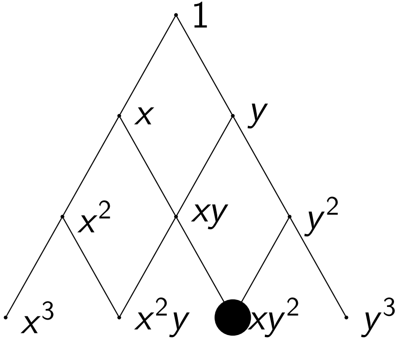
Rowmotion example



$$\mathrm{Row}^2(I) = \langle xy^2, x^4, x^3y, y^4 \rangle$$

$$J = \langle x^4, x^3y, x^2y^2, xy^3, y^4 \rangle$$

Rowmotion example



$$\mathrm{Row}^2(I) = \langle xy^2, x^4, x^3y, y^4 \rangle$$

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Natural base ideals

Let $R = K[x_1, \ldots, x_n]$. There are two natural ideals with respect to which one might apply rowmotion:

1. Powers of the maximal irrelevant ideal: $\mathfrak{m}^d = (x_1, \ldots, x_n)^d$. If $n = 2$, this corresponds to poset rowmotion with respect to the positive root poset for A_d. If $n = 3$, this is a *tetrahedral poset*.
2. Monomial complete intersections: $(x_1^{d_1}, \ldots, x_n^{d_n})$. This corresponds to poset rowmotion with respect to the product of chains $[d_1] \times \cdots \times [d_n]$.

Poset $\leftrightarrow$ Ideal translation

Let I be an artinian monomial ideal, and let P be the poset of standard monomials of I.

1. The height of P is the regularity of R/I.
2. The Hilbert series of R/I is the rank generating function of the dual of P.
3. The cardinality of P is the number of standard monomials of I, or the multiplicity $e(R/I)$ of R/I.

Monomial complete intersections - two variables

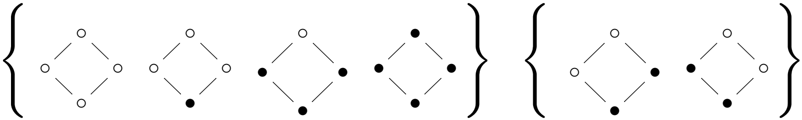
Theorem (Combinatorial theorems: Brower-Schriver (1), S.-Williams (2), Propp-Roby (3-4); Algebraic translation: Cook-S.)

Let $R = K[x, y]$, $d_1, d_2 \geq 1$, *and* $\mathfrak{I} = \{I \mid I \supseteq (x^{d_1}, y^{d_2})\}$.

1. *Rowmotion on the set* $\mathfrak{I}$ *has order* $d_1 + d_2$.
2. *The triple* $(\mathfrak{I}, f(q), \langle Row \rangle)$ *exhibits the* **cyclic sieving phenomenon**, *where* $f(q) := \sum_{I \supseteq (x^{d_1}, y^{d_2})} q^{e(R/I)}$.
3. $e(R/I)$ *is homomesic under the action of rowmotion on* $\mathfrak{I}$ *with average value* $\frac{d_1 d_2}{2}$.
4. *The number of generators of* I *is homomesic under the action of rowmotion on* $\mathfrak{I}$.

Cyclic sieving phenomenon (Reiner-Stanton-White)

Cyclic sieving phenomenon example for $d_1 = d_2 = 2$.



$f(q) = 1 + q + 2q^2 + q^3 + q^4 \qquad \zeta = e^{2\pi i/4} = i$

$f(i^1) = 0$, so 0 elements are fixed under Row^1
$f(i^2) = f(-1) = 2$, so 2 elements are fixed under Row^2
$f(i^3) = f(-i) = 0$, so 0 elements are fixed under Row^3
$f(i^4) = f(1) = 6$, so 6 elements are fixed under Row^4

Monomial complete intersections - two variables

Theorem (Combinatorial theorems: Brower-Schriver (1), S.-Williams (2), Propp-Roby (3-4); Algebraic translation: Cook-S.)

Let $R = K[x, y]$, $d_1, d_2 \geq 1$, *and* $\mathfrak{I} = \{I \mid I \supseteq (x^{d_1}, y^{d_2})\}$.

1. *Rowmotion on the set* $\mathfrak{I}$ *has order* $d_1 + d_2$.
2. *The triple* $(\mathfrak{I}, f(q), \langle Row \rangle)$ *exhibits the cyclic sieving phenomenon, where* $f(q) := \sum_{I \supseteq (x^{d_1}, y^{d_2})} q^{e(R/I)}$.
3. $e(R/I)$ *is* **homomesic** *under the action of rowmotion on* $\mathfrak{I}$ *with average value* $\frac{d_1 d_2}{2}$.
4. *The number of generators of* I *is* **homomesic** *under the action of rowmotion on* $\mathfrak{I}$.

Homomesy (Propp-Roby)

Definition

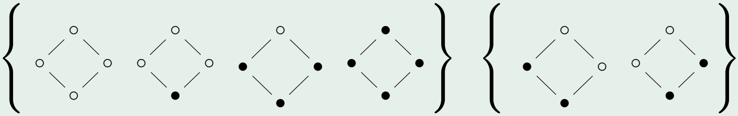
Given a finite *set* S of objects, an invertible *map* $\tau : S \to S$, and a *statistic* $f : S \to \mathbb{Q}$, we say (S, τ, f) exhibits *homomesy* if and only if there exists $c \in \mathbb{Q}$ such that for every τ-orbit $\mathcal{O} \subseteq S$

$$\frac{1}{|\mathcal{O}|} \sum_{x \in \mathcal{O}} f(x) = c.$$

Homomesy (Propp-Roby)

Example

The rowmotion orbits of $J([2] \times [2])$



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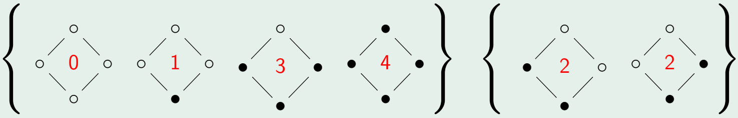
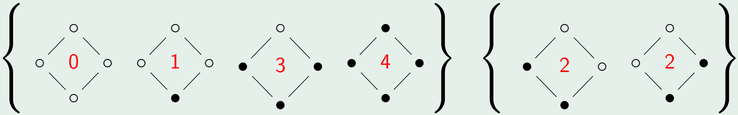


Homomesy (Propp-Roby)

Example

The rowmotion orbits of $J([2] \times [2])$



$$\frac{0+1+3+4}{4} = 2 \qquad\qquad \frac{2+2}{2} = 2$$

Monomial complete intersections - three variables

Theorem (Combinatorial theorems: Cameron-Fon-der-Flaass (1), Rush-Shi (2); S.-Williams (2); Algebraic translation: Cook-S.)

Let $R = K[x, y, z]$, $d_1, d_2 \geq 1$, *and* $\mathfrak{I} = \{I \mid I \supseteq (x^{d_1}, y^{d_2}, z^2)\}$.

1. *Rowmotion on the set* $\mathfrak{I}$ *has order* $d_1 + d_2 + 1$.
2. *The triple* $(\mathfrak{I}, f(q), \langle Row \rangle)$ *exhibits the cyclic sieving phenomenon, where* $f(q) := \sum_{\mathfrak{I}} q^{e(R/I)}$.
3. Conjecture: $e(R/I)$ is homomesic under the action of rowmotion on $\mathfrak{I}$.

Monomial complete intersections - three variables

Theorem (Combinatorial theorem: Dilks-Pechenik-S.; Algebraic translation: Cook-S.)

Let $R = K[x, y, z]$ and $d_1, d_2, d_3 \geq 1$. Then rowmotion on the set $\{I \mid I \supseteq (x^{d_1}, y^{d_2}, z^{d_3})\}$ exhibits **resonance** *with frequency $d_1 + d_2 + d_3 - 1$.*

Definition (Dilks-Pechenik-S.)

Let $G = \langle g \rangle$ be a cyclic group acting on a set X, $\mathcal{C}_\omega = \langle c \rangle$ a cyclic group of order ω acting nontrivially on a set Y, and $f : X \to Y$ a surjection. If $c \cdot f(x) = f(g \cdot x)$ for all $x \in X$, we say the triple (X, G, f) exhibits **resonance** with frequency ω.

Monomial complete intersections - n variables

Theorem (New theorem, inspired by the algebra, proved combinatorially)

Let $R = K[x_1, x_2, \ldots, x_n]$ and $d_1, d_2, \ldots, d_n \geq 1$. Then rowmotion on the set $\{I \mid I \supseteq (x_1^{d_1}, x_2^{d_2}, \ldots, x_n^{d_n})\}$ exhibits **resonance** *with frequency $d_1 + d_2 + \cdots + d_n + 2 - n$.*

Powers of the maximal irrelevant ideal - two variables

Theorem (Combinatorial theorems: Armstrong-Stump-Thomas (1,2,4), S.-Williams (new proof of 2), Hadaddan (3); Algebraic translation: Cook-S.)

Let $R = K[x, y]$, $d \geq 1$, *and* $\mathfrak{I} = \{I \mid I \supseteq (x, y)^d\}$.

1. *Rowmotion on* $\mathfrak{I}$ *has order* $2(d + 1)$ *for* $d \geq 2$ *and order* 2 *for* $d = 1$.
2. *The triple* $(\mathfrak{I}, f(q), \langle Row \rangle)$ *exhibits the cyclic sieving phenomenon, where* $f(q) := \sum_{I \supseteq (x,y)^d} q^{e(R/I)}$.
3. $h(-1)$ *is homomesic under rowmotion on* $\mathfrak{I}$.
4. *The number of generators of* I *is homomesic under rowmotion.*

Thanks!