The Prism Tableau Model for Schubert Polynomials

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Overview

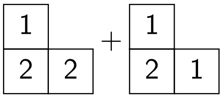
The Prism Tableau Model for Schubert Polynomials

- Describe a tableau based combinatorial model for **Schubert polynomials**
- Give a description of the underlying geometric ideas of the proof
- Apply prism tableaux to study **alternating sign matrix varieties**

The Ring of Symmetric Polynomials

$\Lambda_n = \{f \in \mathbb{Z}[x_1, \ldots, x_n] : w \cdot f = f \text{ for all } w \in S_n\}$

- Schur polynomials $\{s_\lambda\}$ form a $\mathbb{Z}$-linear basis for Λ_n and have applications in geometry and representation theory
- Model for Schur polynomials as a sum over **semistandard Young tableaux**



$$s_{(2,1)}(x_1, x_2) = x_1 x_2^2 + x_1^2 x_2$$

There is an inclusion:

$$\Lambda_n \hookrightarrow \texttt{Pol} = \mathbb{Z}[x_1, x_2, ...]$$

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Question: How do we lift the Schur polynomials to a basis of `Pol`?

An answer: Schubert polynomials

Schubert Polynomials

Introduced by Lascoux and Schützenberger in 1982 to study the cohomology of the **complete flag variety**

- Indexed by permutations, $\{\mathfrak{S}_w : w \in S_n\}$

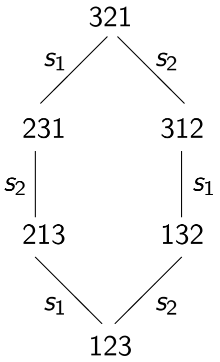
To find $\mathfrak{S}_w$:

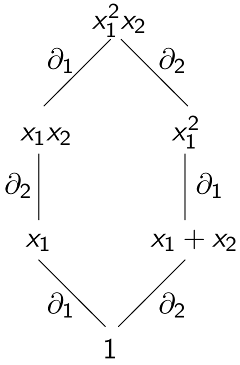
- $\mathfrak{S}_{w_0} := x_1^{n-1} x_2^{n-2} \ldots x_{n-1}$
- The rest are defined recursively by **divided difference operators:**

$$\partial_i f := \frac{f - s_i \cdot f}{x_i - x_{i+1}}$$

- $\mathfrak{S}_{ws_i} := \partial_i \mathfrak{S}_w$ if $w(i) > w(i+1)$

Schubert Polynomials for S_3





The Schubert Basis

Schubert polynomials as a basis:

- There is a natural inclusion of symmetric groups $S_n \xrightarrow{\iota} S_{n+1}$
- Schubert polynomials are **stable** under this inclusion:

$$\mathfrak{S}_w = \mathfrak{S}_{\iota(w)}$$

- $\{\mathfrak{S}_w : w \in S_\infty\}$ forms a $\mathbb{Z}$-linear basis of $\texttt{Pol} = \mathbb{Z}[x_1, x_2, \ldots]$

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Schubert polynomials as a lift of Schur polynomials:

- Every Schur polynomial is a Schubert polynomial for some $w \in S_\infty$
- $\mathfrak{S}_w$ is a Schur polynomial if and only if w is **Grassmannian**

Problem: Is there a combinatorial model for $\mathfrak{S}_w$ that is analogous to semistandard tableau?

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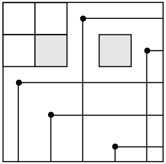
A new solution: Prism Tableaux

What is a Prism Tableau?

Some Definitions



$w = 35124$

Each permutation has an associated:

- **diagram**:

$$D(w) = \{(i,j) : 1 \leq i,j \leq n, w(i) > j \text{ and } w^{-1}(j) > i\} \subset n \times n$$

- **essential set**:

$$\mathcal{E}ss(w) = \{\text{southeast-most boxes of each component of } D(w)\}$$

- **rank function**: $r_w(i,j) =$ the rank of the (i,j) NW submatrix

The Shape

Fix $w \in S_n$.

- Each $e = (a, b) \in \mathcal{E}ss(w)$ indexes a color
- Let R_e be an $(a - r_w(e)) \times (b - r_w(e))$ rectangle in the $n \times n$ grid (left justified, bottom row in same row as e)
- Define the **shape**:

$$\lambda(w) = \bigcup_{e \in \mathcal{E}ss(w)} R_e$$

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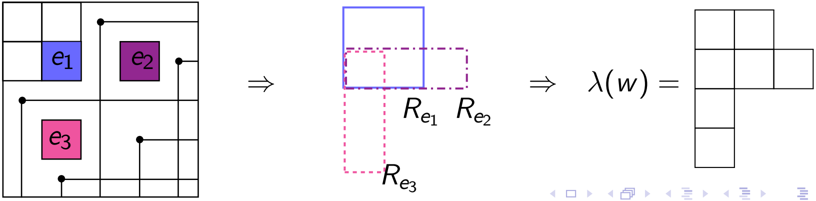
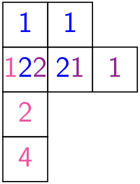
Example: $w = 35142$



Prism Tableaux

A **prism tableau** for w is a filling of $\lambda(w)$ with colored labels, indexed by $\mathcal{E}ss(w)$ so that labels of color e:

- sit in boxes of R_e
- weakly decrease along rows from left to right
- strictly increase along columns from top to bottom
- are **flagged** by row: no label is bigger than the row it sits in



The Weight of a Tableau

Define the weight:

$$\texttt{wt}(T) = \prod_i x_i^{\#\text{ of antidiagonals containing a label of number } i}$$

Example:

$$T = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad\qquad \texttt{wt}(T) = x_1^2 x_2$$

Minimal Prism Tableaux

Fix a prism tableau T.

- T is **minimal** if the degree of $\texttt{wt}(T) = \ell(w)$.

Example: $w = 1432$

11	1
3	

$\texttt{wt}(T) = x_1^2 x_3$

minimal

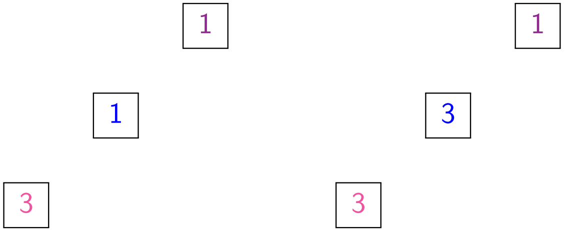
21	1
3	

$\texttt{wt}(T) = x_1^2 x_2 x_3$

not minimal

Unstable Triples

- We say labels $(\ell_c, \ell_d, \ell'_e)$ in the same antidiagonal T form an **unstable triple** if $\ell < \ell'$ and the tableau T' obtained by replacing ℓ_c with ℓ'_c is itself a prism tableau.



The Prism model for Schubert Polynomials

Let $\texttt{Prism}(w)$ be the set of **minimal prism tableaux** for w which have **no unstable triples**.

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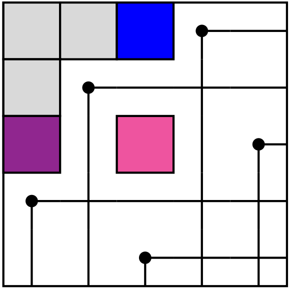
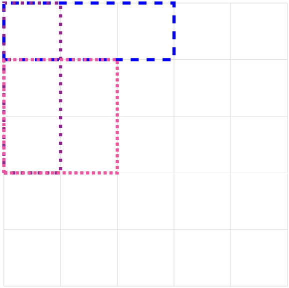
Theorem (W.- A. Yong 2015)

$$\mathfrak{S}_w = \sum_{T \in \texttt{Prism}(w)} \texttt{wt}(T).$$

Example for $w = 42513$





Example for $w = 42513$ (continued)

11	1	1
22	1	
33	3	

In $\texttt{Prism}(w)$

11	1	1
22	1	
33	2	

In $\texttt{Prism}(w)$

11	1	1
22	2	
33	3	

Not minimal

11	1	1
21	1	
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Unstable triple

11	1	1
21	1	
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Unstable triple

11	1	1
21	1	
32	2	

Not minimal

Example for $w = 42513$ (continued)

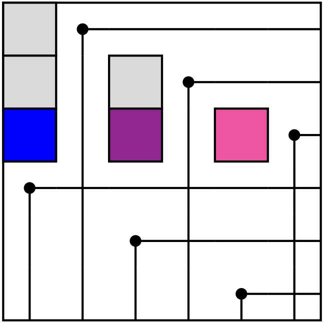
$$\mathfrak{S}_{42513} = \begin{array}{|c|c|c|}\hline 11 & 1 & 1 \\ \hline 22 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} 33 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} + \begin{array}{|c|c|c|}\hline 11 & 1 & 1 \\ \hline 22 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} 33 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$$

$$= x_1^3 x_2 x_3^2 + x_1^3 x_2^2 x_3$$

Grassmannian Permutations

When w is Grassmannian, all essential boxes lie in the same row.

Example: $w = 246135$.



Specialization to Schur Polynomials

$$\mathfrak{S}_{246135} = \begin{array}{|c|c|c|}\hline 1 \\ \cline{1-2} 22 & 2 \\ \hline 333 & 33 & 1 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 1 \\ \cline{1-2} 22 & 2 \\ \hline 333 & 33 & 2 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 1 \\ \cline{1-2} 22 & 2 \\ \hline 333 & 33 & 3 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 1 \\ \cline{1-2} 22 & 1 \\ \hline 333 & 22 & 1 \\ \hline\end{array} + \begin{array}{|c|c|c|}\hline 1 \\ \cline{1-2} 22 & 1 \\ \hline 333 & 22 & 2 \\ \hline\end{array} +$$

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Specialization to Schur Polynomials

$$\mathfrak{S}_{246135} = \begin{array}{lll} 1 & & \\ 22 & 2 & \\ 333 & 33 & 1 \end{array} + \begin{array}{lll} 1 & & \\ 22 & 2 & \\ 333 & 33 & 2 \end{array} + \begin{array}{lll} 1 & & \\ 22 & 2 & \\ 333 & 33 & 3 \end{array} + \begin{array}{lll} 1 & & \\ 22 & 1 & \\ 333 & 22 & 1 \end{array} + \begin{array}{lll} 1 & & \\ 22 & 1 & \\ 333 & 22 & 2 \end{array} + \begin{array}{lll} 1 & & \\ 22 & 1 & \\ 333 & 33 & 1 \end{array} + \begin{array}{lll} 1 & & \\ 22 & 1 & \\ 333 & 33 & 2 \end{array} + \begin{array}{lll} 1 & & \\ 22 & 1 & \\ 333 & 33 & 3 \end{array}$$

Forgetting colors gives the following expansion of the Schur polynomial:

$$s_{(3,2,1)} = \begin{array}{lll} 1 & & \\ 2 & 2 & \\ 3 & 3 & 1 \end{array} + \begin{array}{lll} 1 & & \\ 2 & 2 & \\ 3 & 3 & 2 \end{array} + \begin{array}{lll} 1 & & \\ 2 & 2 & \\ 3 & 3 & 3 \end{array} + \begin{array}{lll} 1 & & \\ 2 & 1 & \\ 3 & 2 & 1 \end{array} + \begin{array}{lll} 1 & & \\ 2 & 1 & \\ 3 & 2 & 2 \end{array} + \begin{array}{lll} 1 & & \\ 2 & 1 & \\ 3 & 3 & 1 \end{array} + \begin{array}{lll} 1 & & \\ 2 & 1 & \\ 3 & 3 & 2 \end{array} + \begin{array}{lll} 1 & & \\ 2 & 1 & \\ 3 & 3 & 3 \end{array}$$

The Geometry Behind the Model

Gröbner Geometry of Schubert Polynomials (A. Knutson - E. Miller 2005)

Identify the **flag variety** with $B_- \backslash GL_n$.

$$\begin{array}{ccc} GL_n & \xrightarrow{\iota} & \mathrm{Mat}_{n\times n} \\ {\scriptstyle \pi}\big\downarrow & & \\ B_- \backslash GL_n & & \end{array} \qquad\qquad \begin{array}{ccc} \pi^{-1}(\mathfrak{X}) & \xrightarrow{\iota} & X \\ {\scriptstyle \pi}\big\downarrow & & \\ \mathfrak{X} & & \end{array}$$

For $\mathfrak{X} \subset B_- \backslash GL_n$, there is a corresponding $X = \overline{\iota(\pi^{-1}(\mathfrak{X}))}$.

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For $\mathfrak{X} \subset B_- \backslash GL_n$, there is a corresponding $X = \overline{\iota(\pi^{-1}(\mathfrak{X}))}$.

- X is T stable, so $[X]_T \in H_T(\mathrm{Mat}_{n\times n}) \cong \mathbb{Z}[x_1, \ldots x_n]$.
- $[X]_T$ is a coset representative for $[\mathfrak{X}]$ in $H^*(B_- \backslash GL_n) \cong \mathbb{Z}[x_1, \ldots, x_n]/I^{S_n}$

The cohomology of the flag variety has an additive basis in terms of **Schubert classes**, defined by **Schubert varieties** $\mathfrak{X}_w$.

The polynomials $[X_w]_T$ represent the Schubert classes

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Theorem (Knutson-Miller '05)

$$[X_w]_T = \mathfrak{S}_w$$

$X_w := \overline{\iota(\pi^{-1}(\mathfrak{X}))}$ is called a **matrix Schubert variety** and has an explicit description in terms of "at most" rank conditions on $\mathrm{Mat}_{n \times n}$.

Using **Gröbner degeneration**, we associate to X a collection of objects called **plus diagrams**, which index **coordinate subspaces** of $\mathtt{init}_\prec X$.

When the degeneration is reduced,

$$[X]_T = [\mathtt{init}_\prec X]_T \quad = \sum_{\mathcal{P} \in \mathtt{MinPlus}(X)} \mathtt{wt}(\mathcal{P})$$

For a matrix Schubert variety X_w,

$$\mathtt{MinPlus}(X_w) \leftrightarrow \text{ Pipe Dreams / RC Graphs for } w.$$

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$$\mathtt{MinPlus}(X_w) \leftrightarrow \text{ Pipe Dreams / RC Graphs for } w.$$

Slogan: In "nice" situations, if you know the plus diagrams, you know the T-equivariant class.

Back to Prism Tableaux

A left-justified rectangle R determines a unique **biGrassmannian** permutation $u \in S_\infty$.

Fillings of R biject with minimal **plus diagrams** for u and the bijection is **weight preserving**.

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & 2 \\ \hline \end{array} \quad \mapsto \quad \begin{bmatrix} \cdot & \cdot & + & \cdot & \cdot \\ + & \cdot & + & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Prism Tableau biject with $\mathcal{E}ss(w)$-tuples of minimal plus diagrams.

If $\texttt{biGrass}(w) = \{u_1, \ldots, u_k\}$ then

$$X_w = X_{u_1} \cap \ldots \cap X_{u_k}.$$

Furthermore,

$$\texttt{init}_{\prec} X_w = \texttt{init}_{\prec} X_{u_1} \cap \ldots \cap \texttt{init}_{\prec} X_{u_k}.$$

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Note 1: Not all unions are minimal.

Note 2: May be many different ways to produce the same plus diagram for A.

Many ways to produce the same plus diagram

Let $w = 42513$. Then $\texttt{biGrass}(w) = \{41235, 23415, 14523\}$.

$$\mathcal{P} = \begin{bmatrix} + & + & + & \cdot & \cdot \\ + & \cdot & + & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\left\{ \begin{bmatrix} ++ & ++ & ++ & \cdot & \cdot \\ + & \cdot & + & \cdot & \cdot \\ ++ & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad \begin{bmatrix} ++ & + & ++ & \cdot & \cdot \\ ++ & \cdot & + & \cdot & \cdot \\ ++ & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \right\}.$$

Many ways to produce the same plus diagram

Let $w = 42513$. Then $\texttt{biGrass}(w) = \{{\color{blue}41235}, {\color{purple}23415}, {\color{magenta}14523}\}$.

$$\mathcal{P} = \begin{bmatrix} + & + & + & \cdot & \cdot \\ + & \cdot & + & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\left\{ \begin{bmatrix} {\color{purple}+}{\color{blue}+} & {\color{blue}+}{\color{magenta}+} & {\color{blue}+}{\color{magenta}+} & \cdot & \cdot \\ {\color{purple}+} & \cdot & {\color{magenta}+} & \cdot & \cdot \\ {\color{purple}+}{\color{magenta}+} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad \begin{bmatrix} {\color{purple}+}{\color{blue}+} & {\color{blue}+} & {\color{blue}+}{\color{magenta}+} & \cdot & \cdot \\ {\color{purple}+}{\color{magenta}+} & \cdot & {\color{magenta}+} & \cdot & \cdot \\ {\color{purple}+}{\color{magenta}+} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \right\}.$$

When $\mathcal{P}$ is minimal, tuples of plus diagrams which have **support** $\mathcal{P}$ form a **lattice**.

Alternating Sign Matrices

Prism Tableaux for Alternating Sign Matrix Ideals

A matrix A is an **alternating sign matrix** (ASM) if:

- A has entries in $\{-1, 0, 1\}$
- Rows and columns sum to 1
- Non-zero entries alternate in sign along rows and columns

Let ASM(n) be the set of $n \times n$ ASMs.

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in \mathrm{ASM}(4)$$

Background on ASMs

The Alternating Sign Matrix Conjecture (Mills, Robbins, and Rumsey '83):

$$\#\text{ASM}(n) = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

- Proved by independently by Zielberger and Kuperberg in 1995.

ASM(n) forms the **Dedekind-MacNeille completion** of the Bruhat order on S_n (Lascoux-Schützenberger, '96).

ASM Varieties

To $A \in \mathrm{ASM}(n)$ there is an associated **ASM Variety** X_A.

- Defined by "at most" rank conditions on $\mathrm{Mat}_{n \times n}$ coming from the **corner sum matrix** associated to A.
- When A is a permutation matrix, X_A is a matrix Schubert variety.
- X_A is T stable.

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- X_A is T stable.

Question: What is $\mathfrak{S}_A := [X_A]_T$?

Partial Answer: When A is a permutation matrix, $\mathfrak{S}_A$ is a Schubert polynomial. (Knutson-Miller '05).

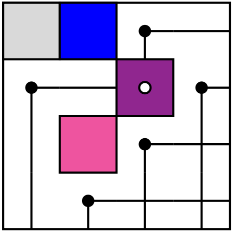
We may associate to A a shape. Let $\texttt{Prism}(A)$ be the set of minimal prism tableaux for A with no unstable triples.

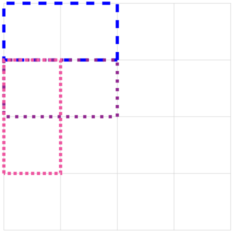
Theorem (W.- A. Yong)

$$\mathfrak{S}_A = \sum_{T \in \texttt{Prism}(A)} \texttt{wt}(T)$$

The Diagram and Shape of an ASM

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$





$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

1	1
11	1
2	

+

1	1
11	1
3	

+

1	1
21	2
2	

$$\mathfrak{S}_A = x_1^3 x_2 + x_1^3 x_3 + x_1^2 x_2^2$$

Thank You!