Algebraic Applications of the Theory of Violator Spaces

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Violator spaces: Definition and example

Definition (Gärtner et al, 2008)

A violator space is a pair (H, V), where H is a finite set and $V : 2^H \to 2^H$ is a mapping such that:

1. *For all $G \subseteq H$, $G \cap V(G) = \emptyset$ (consistency)*
2. *For all $F \subseteq G \subseteq H$, such that $G \cap V(F) = \emptyset$, $V(G) = V(F)$ (locality)*

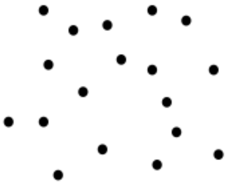
- The mapping V associates to every subset $G \subseteq H$ the set of things in H that "violate" G
- Think of H as a set of constraints
- Get to choose what "violates" means for your particular problem
- Examples: LP-type problems, geometric optimization problems, smallest enclosing ball problem

Smallest enclosing ball in $\mathbb{R}^2$:

Problem: Given a set of points in $\mathbb{R}^2$, find the smallest circle containing them.

Setup:

- H, a set of points $\mathbb{R}^2$
- V: For $G \subset H$, a point p outside of G violates G if adding p to G increases the size of the smallest circle containing G.



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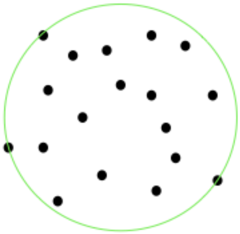

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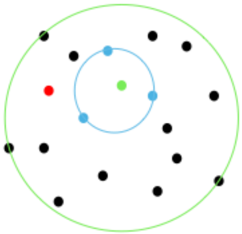

- $G =$ blue points
- Red point violates G
- Green point does not violate G

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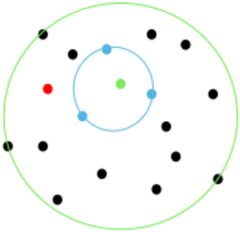

Key observation:

At most 3 points of H determine the unique smallest circle containing H

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Definition (Gärtner et al, 2008)

A basis of a violator space (H, V) is a subset $B \subseteq H$ such that $B \cap V(F) \neq \emptyset$ holds for all proper subsets $F \subsetneq B$. The combinatorial dimension is the size of the largest basis for (H, V).

What does this buy you?

Key idea:

Violator spaces provide an abstract framework for formulating many types of optimization problems which is useful for designing efficient algorithms.

Clarkson's algorithm (Clarkson, 1995):

A randomized algorithm that performs biased sampling to find a basis.
Input: (H, V); δ, the combinatorial dimension
Output: $\mathcal{B}$, a basis for H
Given a violator space (H, V), some subset $G \subsetneq H$, and some elements $h \in H \setminus G$, the primitive test decides whether $h \in V(G)$.

Theorem (Clarkson, 1995; Škovroň, 2007)

Clarkson's algorithm finds a basis $\mathcal{B}$ for (H, V) in an expected $O(\delta|H| + \delta^{O(\delta)})$ calls to the primitive.

Goal:

Take problems from computational algebra and fit them into the framework of violator spaces.

Each potential application requires three ingredients:

- The right notion of "violates"
- A bound on δ, the combinatorial dimension
- A primitive test

Overdetermined systems of polynomials

Problem

Suppose that $\{f_1, \ldots, f_s\}$ *(*$s \gg 0$*) is a collection of polynomials in* n *variables and we are interested in solving the system* $f_1 = \cdots = f_s = 0$*.*

The ingredients (De Loera-Petrović-Stasi, 2015):

- Violator: $H = \{f_1, \ldots, f_s\}$; if $G \subset H$, f_i violates G if f_i does not vanish on the variety $\mathcal{V}(G)$.
- Combinatorial dimension: rank of coefficient matrix:

$$\delta = rank \left(\begin{array}{cc} & \textit{monomials} \\ \begin{array}{c} f_1 \\ \vdots \\ f_s \end{array} & \left(\begin{array}{c} \\ \textit{coefficients} \\ \\ \end{array} \right) \end{array} \right).$$

- Primitive test: GB calculation

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Example (Mayr-Meyer Ideal:)

The Mayr-Meyer ideal J(n,d) is an ideal in $10n + d$ variables where the minimal generators have degree $d + 2$. It is a pathological example that is not to achieve the doubly exponential bound in n for GB computations. In the case $n = d = 2$, we added two polynomials to the 24 minimal generators of $J(2, 2)$ to make the system infeasible. Using a prototype for V_{solve} in `Macaulay2`, we found a basis of size 2 an average of *8 seconds*. The Gröbner computation on the same machine lasted 18+ hours without terminating.

Small generating sets and semi-algebraic sets

Theorem (De Loera-Petrović-Stasi, 2015)

There exists a violator $V_{SmallGen}$ for finding small generating sets of homogenous ideals in a polynomials ring.

Theorem (De Loera-Petrović-Stasi-W., 2016+)

There exists a violator $V_{SemiAlg}$ for finding minimal representations of elementary semi-algebraic sets.

Current work

What's happening next:

- Showing that the violators V_{Solve}, V_{SmallGen}, and V_{SemiAlg} satisfy addition properties in the violator framework
- Extending to other problems in computation algebra
- Finding nice applications

References

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[2] Clarkson, K.L., 1995. Las Vegas algorithms for linear and integer programming. J. ACM 42 (2), 488-499.

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[4] De Loera, J.A., Petrović, P., Stasi, D. 2015. Random Sampling in Computational Algebra: Helly Numbers and Violator Spaces. J. Symb. Comp. 10.1016/j.jsc.2016.01.001. arXiv:1503.08804.