On n-pseudo valuation domains (n-PVD) (preliminary talk)

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Thanks for coming,(**History and References)**

(1) J. R. Hedstrom and E. G. Houston, Pseudo-valuation domains. Pacific J. Math. 4(1978), 551–567.

(2) J. R. Hedstrom and E. G. Houston, Pseudo-valuation domains, II. Houston J. Math.4(1978), 199–207.

Pseudo valuation domains were born in 1978 (Hedtrom-Houston) : R is an integral domain with quotient field K, A prime ideal Q of R is called strongly prime (1-powerful semiprimary ideal) if whenever $xy \in Q$ for some $x, y \in K$, then $x \in Q$ or $y \in Q$. If every prime ideal of R is strongly prime, then R is called a pseudo-valuation domain. (Many authors studied this class of domains including myself).

The concept of pseudo valuation domains is a generalization of the concept of valuation domains: R is called an integral domain with quotient field K, then R is called a valuation domain if $x \in R$ or $x^{-1} \in R$ for every nonzero $x \in K$.

(3) D. D. Anderson and M. Zafrullah, Almost Bezout domains, J. Algebra, 142(1991), 285–309.

(4) D. D. Anderson and M. Zafrullah, Almost Bezout domains, III,

Bull. Math. Soc. Sci. Math. Roumanie, 51(2008), 3–9.
Almost valuation domains were born in 1991 and 2008 and (D.D. Anderson-Zafrulla): R is an integral domain with quotient field K, then R is called an almost valuation domain if for every nonzero $x \in K$, there exists n (n depends on x) such that $x^n \in R$ or $x^{-n} \in R$. So every valuation domain is an almost valuation domain.
For a recent article on almost valuation domains see
[5] N. Mahdou, A. Mimouni and M. Moutui, On almost valuation and almost Bezout rings, Commun. Algebra, 43(2015), 297–308.
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(6) A. Badawi, On pseudo-almost valuation domains, Commun. Algebra 35(2007), 1167–1181.
Pseudo-almost valuation domain was born in 2007 (Badawi): Let R be an integral domain with quotient field K, I be a prime ideal of R. We say that I is a *pseudo-strongly prime ideal* of R if whenever $x^n y^n \in I$ for
some $x, y \in K$, then $x^n \in I$ or $y^n a \in I$ for every $a \in I$. If every prime ideal of R is a pseudo-prime ideal of R, then R is called a

pseudo-almost valuation domain (It turns out that every almost valuation domain and pseudo-valuation domain is a pseudo-almost valuation domain but not vice-versa).

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The following implications hold but none of them is revisable ...
Valuation domain $\Rightarrow$ Pseudo-valuation domain $\Rightarrow$ pseudo almost valuation domain.
valuation domain $\Rightarrow$ almost-valuation domain $\Rightarrow$ Almost-pseudo almost valuation domain.
Now we will introduce n-pseudo valuation domain So we will have the following implications hold but none of them is revisable
Valuation domain $\Rightarrow$ Pseudo-valuation domain $\Rightarrow$ Almost-pseudo almost valuation domain $\Rightarrow$ n-pseudo valuation domain
valuation domain $\Rightarrow$ almost valuation domain $\Rightarrow$ n-pseudo valuation domain

n-powerful semiprimary ideals of integral domains

Definition

Let R be an integral domain with quotient field K, I be a proper ideal of R and $n \geq 1$ be a positive integer. We say that I is a n-powerful ideal of R if whenever $x^n y^n \in I$ for some $x, y \in K$, then $x^n \in R$ or $y^n \in R$. We say I is a n-powerful semiprimary ideal of R if whenever $x^n y^n \in I$ for some $x, y \in K$, then $x^n \in I$ or $y^n \in I$.

Example

Let $R = Q + X^2C + X^4C[[X]]$, where Q is the field of rational numbers and C is the field of complex numbers. Then one can see that R is neither a PAVD as in Badawi nor a PVD as in Hedstrom-Houston nor an almost valuation domain as in Anderson-Zafrulla. However, it is easily checked that R is a 4-PVD with maximal ideal $M = X^2C + X^4C[[X]]$ and $\overline{R} = \overline{Q} + XC[[X]]$ is a PVD with maximal ideal $N = \{x \in K \mid x^n \in M\} = XC[[X]]$, where $\overline{Q}$ is the algebraic closure of Q inside C, and K is the quotient field of R. Note that $\overline{R}$ is not a valuation domain and R is not an n-PVD for every $1 \leq n \leq 3$.

Theorem

Let $n \geq 1$ and I be a prime ideal of an integral domain R with quotient field K. Then I is a n-powerful semiprimary ideal of R if and only if I is a n-powerful ideal of R.

Theorem

Assume $P \subseteq Q$ are prime ideals of an integral domain R. If Q is a n-powerful semiprimary ideal of R for some positive integer $n \geq 1$, then P is a n-powerful semiprimary ideal of R.

Definition

Let R be an integral domain with quotient field K and $n \geq 1$ be a positive integer. We say that R is an n-pseudo valuation domain (n-PVD) if every prime ideal of R is a n-powerful semiprimary ideal of R. Note that if $n = 1$, then a pseudo-valuation domain in the sense of Houston-Hedstrom is a 1-PVD.

Throughout this section R denotes an integral domain with quotient field K.

Theorem

Let $n \geq 1$ and a assume that R is an n-PVD. Then R is a quasilocal domain.

Corollary

Let $n \geq 1$ be a positive integer. Then an integral domain R is an n-PVD if and only if a maximal ideal of R is a n-powerful semiprimary ideal of R if and only if a maximal ideal of R is a n-powerful ideal of R.

Definition

Let R be a commutative ring and $n \geq 1$ be a positive integer. A prime ideal P of R is called an n-divided prime ideal of R if $x^n \mid p^n$ (in R) for every $x \in R \setminus P$ and for every $p \in P$. A commutative ring R is called an n-divided ring if every prime ideal of R is an n-divided prime ideal of R. Note that if $n = 1$, then a divided ring in the sense of Dobbs-Badawi is a 1-divided ring.

Corollary

Assume that an integral domain R is an n-PVD for some positive integer $n \geq 1$. Then R is an n-divided domain and the set of all prime ideals of R are linearly ordered by inclusion.

Definition

Let $n \geq 1$ be a positive integer and S be a subset of an integral domain R with quotient field K. Then $E_n(S) = \{x \in K \mid x^n \notin S\}$ and $A_n(S) = \{x^n \mid x^n \in S, x \in K\}$.

Theorem

Let $n \geq 1$, R be an integral domain, and P be a prime ideal of R. Then P is a n-powerful semiprimary ideal of R if and only if $x^{-n}d \in P$ for every $x \in E_n(P)$ and every $d \in A_n(P)$.

Corollary

Let $n \geq 1$ and R be a quasilocal domain with maximal ideal M. The following statements are equivalent.

1. *R is an n-PVD.*
2. *$x^{-n}d \in R$ for every $x \in E_n(R)$ and every $d \in A_n(M)$.*
3. *$x^{-n}d \in M$ for every $x \in E_n(M)$ and every $d \in A_n(M)$.*

Theorem

Let $n \geq 1$ and R be a root closed integral domain with quotient field K. Then R is a PVD if and only if R is an n-PVD.

We recall from Anderson-Zafrulla that an integral domain R with quotient filed K is called an *almost valuation domain* if for every nonzero $x \in K$, there is an integer $n \geq 1$ (n depends on x) such that $x^n \in R$ or $x^{-n} \in R$. We have the following definition.

Definition

Let $n \geq 1$ be a positive integer and R be an integral domain with quotient filed K is called an n-valuation domain (n-VD) if for every nonzero $x \in K$, we have $x^n \in R$ or $x^{-n} \in R$. It is clear that an n-valuation domain is an almost valuation domain.

Theorem

Let R be an n-PVD for some positive integer $n \geq 1$ with maximal ideal M. Suppose that V is an overring of R such that $\frac{1}{s} \in V$ for some $s \in M$. Then V is an n-VD (and hence V is an almost valuation domain).

Theorem

Let R be an n-PVD for some positive integer $n \geq 1$ with maximal ideal M. Suppose that P is a prime ideal of R such that $P \neq M$. Then R_P is an n-VD (and hence R_P is an almost valuation domain). Furthermore, $x^n \in R$ for every $x \in P_P$, and hence $P_P \subset \overline{R}$.

Theorem

Suppose that an integral domain R with quotient field K admits a principal prime ideal P of R that is an n-divided ideal of R for some positive integer $n \geq 1$, then P is a maximal ideal of R. In particular, if P is an n-powerful semiprimary ideal of R for some positive integer $n \geq 1$, then P is a maximal ideal of R and R is an n-VD.

Theorem

Let $n \geq 1$, R be an integral domain with quotient field K and P be a prime ideal of R . Assume that P is an n-powerful semiprimary ideal of R. Then P is an mn-powerful semiprimary ideal of R for every integer $m \geq 1$. Furthermore, if $x^{mn} \in P$ for some integer $m \geq 1$ and $x \in K$, then $x^n \in P$. In particular, if R is an n-PVD, then R is an mn-PVD for every integer $m \geq 1$.

Theorem

Let $n \geq 1$ be an integer and R be an n-PVD with maximal ideal M and with quotient field K. Assume that B is overring of R that is integral over R. Then B is an n-PVD with maximal ideal $\sqrt{MB} = \{x \in B \mid x^n \in M\}$.

Theorem

Let $n \geq 1$ be an integer and R be an n-PVD with maximal ideal M and with quotient field K. Then $\overline{R}$ is a PVD (1-PVD) with maximal ideal $N = \{x \in K \mid x^n \in M\}$.

Theorem

Let $n \geq 1$ be an integer and R be a quasilocal domain with maximal ideal M and with quotient field K. Then R is an n-PVD if and only if $\overline{R}$ is a PVD with maximal ideal $N = \{x \in K \mid x^n \in M\}$.

Corollary

Let $n \geq 1$ be an integer and R be a quasilocal domain with maximal ideal M and with quotient field K. The following statements are equivalent.

1. *R is an n-PVD.*
2. *$\overline{R}$ is a PVD with maximal ideal $N = \{x \in K \mid x^n \in M\}$.*
3. *$N = \{x \in K \mid x^n \in M\}$ is a maximal ideal of $\overline{R}$ such that $(N : N)$ is a valuation domain with maximal ideal N.*

Theorem

Let $n \geq 1$ be an integer and R be an n-PVD with maximal ideal M and with quotient field K. Then every overring of R is an n-PVD if and only if $\overline{R}$ is a valuation domain.