On $n$-pseudo valuation domains ($n$-PVD) (preliminary talk)

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Thanks for coming,


Pseudo valuation domains were born in 1978 (Hedstrom-Houston): $R$ is an integral domain with quotient field $K$, a prime ideal $Q$ of $R$ is called strongly prime (1-powerful semiprimary ideal) if whenever $xy \in Q$ for some $x, y \in K$, then $x \in Q$ or $y \in Q$. If every prime ideal of $R$ is strongly prime, then $R$ is called a pseudo-valuation domain. (Many authors studied this class of domains including myself).

The concept of pseudo valuation domains is a generalization of the concept of valuation domains: $R$ is called an integral domain with quotient field $K$, then $R$ is called a valuation domain if $x \in R$ or $x^{-1} \in R$ for every nonzero $x \in K$.


(4) D. D. Anderson and M. Zafrullah, Almost Bezout domains, III,

Ayman Badawi American University of Sharjah, Sharjah, UAE On $n$-pseudo valuation domains ($n$-PVD) (preliminary talk)
Almost valuation domains were born in 1991 and 2008 and (D.D. Anderson-Zafrulla): $R$ is an integral domain with quotient field $K$, then $R$ is called an almost valuation domain if for every nonzero $x \in K$, there exists $n$ ($n$ depends on $x$) such that $x^n \in R$ or $x^{-n} \in R$. So every valuation domain is an almost valuation domain.


Pseudo-almost valuation domain was born in 2007 (Badawi): Let $R$ be an integral domain with quotient field $K$, $I$ be a prime ideal of $R$. We say that $I$ is a pseudo-strongly prime ideal of $R$ if whenever $x^n y^n \in I$ for some $x, y \in K$, then $x^n \in I$ or $y^n a \in I$ for every $a \in I$. If every prime ideal of $R$ is a pseudo-prime ideal of $R$, then $R$ is called a
pseudo-almost valuation domain (It turns out that every almost valuation domain and pseudo-valuation domain is a pseudo-almost valuation domain but not vice-versa).

The following implications hold but none of them is revisable ...
Valuation domain $\Rightarrow$ Pseudo-valuation domain $\Rightarrow$ pseudo almost valuation domain.
valuation domain $\Rightarrow$ almost-valuation domain $\Rightarrow$ Almost-pseudo almost valuation domain.
Now we will introduce $n$-pseudo valuation domain So we will have the following implications hold but none of them is revisable
Valuation domain $\Rightarrow$ Pseudo-valuation domain $\Rightarrow$ Almost-pseudo almost valuation domain $\Rightarrow$ $n$-pseudo valuation domain
valuation domain $\Rightarrow$ almost valuation domain $\Rightarrow$ $n$-pseudo valuation domain

$n$-powerful semiprimary ideals of integral domains
### Definition

Let $R$ be an integral domain with quotient field $K$, $I$ be a proper ideal of $R$ and $n \geq 1$ be a positive integer. We say that $I$ is a $n$-powerful ideal of $R$ if whenever $x^ny^n \in I$ for some $x, y \in K$, then $x^n \in R$ or $y^n \in R$. We say $I$ is a $n$-powerful semiprimary ideal of $R$ if whenever $x^ny^n \in I$ for some $x, y \in K$, then $x^n \in I$ or $y^n \in I$.

### Example

Let $R = \mathbb{Q} + X^2\mathbb{C} + X^4\mathbb{C}[[X]]$, where $\mathbb{Q}$ is the field of rational numbers and $\mathbb{C}$ is the field of complex numbers. Then one can see that $R$ is neither a PAVD as in Badawi nor a PVD as in Hedstrom-Houston nor an almost valuation domain as in Anderson-Zafrulla. However, it is easily checked that $R$ is a 4-PVD with maximal ideal $M = X^2\mathbb{C} + X^4\mathbb{C}[[X]]$ and $\overline{R} = \overline{\mathbb{Q}} + X\mathbb{C}[[X]]$ is a PVD with maximal ideal $N = \{ x \in K \mid x^n \in M \} = X\mathbb{C}[[X]]$, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$, and $K$ is the quotient field of $R$. Note that $\overline{R}$ is not a valuation domain and $R$ is not an $n$-PVD for every $1 \leq n \leq 3$. 
Theorem
Let \( n \geq 1 \) and \( I \) be a prime ideal of an integral domain \( R \) with quotient field \( K \). Then \( I \) is a \( n \)-powerful semiprimary ideal of \( R \) if and only if \( I \) is a \( n \)-powerful ideal of \( R \).

Theorem
Assume \( P \subseteq Q \) are prime ideals of an integral domain \( R \). If \( Q \) is a \( n \)-powerful semiprimary ideal of \( R \) for some positive integer \( n \geq 1 \), then \( P \) is a \( n \)-powerful semiprimary ideal of \( R \).

Definition
Let \( R \) be an integral domain with quotient field \( K \) and \( n \geq 1 \) be a positive integer. We say that \( R \) is an \( n \)-pseudo valuation domain (\( n \)-PVD) if every prime ideal of \( R \) is a \( n \)-powerful semiprimary ideal of \( R \). Note that if \( n = 1 \), then a pseudo-valuation domain in the sense of Houston-Hedstrom is a 1-PVD.

Throughout this section \( R \) denotes an integral domain with quotient field \( K \).
Theorem

Let \( n \geq 1 \) and assume that \( R \) is an \( n \)-PVD. Then \( R \) is a quasilocal domain.

Corollary

Let \( n \geq 1 \) be a positive integer. Then an integral domain \( R \) is an \( n \)-PVD if and only if a maximal ideal of \( R \) is a \( n \)-powerful semiprimary ideal of \( R \) if and only if a maximal ideal of \( R \) is a \( n \)-powerful ideal of \( R \).

Definition

Let \( R \) be a commutative ring and \( n \geq 1 \) be a positive integer. A prime ideal \( P \) of \( R \) is called an \( n \)-divided prime ideal of \( R \) if \( x^n | p^n \) (in \( R \)) for every \( x \in R \setminus P \) and for every \( p \in P \). A commutative ring \( R \) is called an \( n \)-divided ring if every prime ideal of \( R \) is an \( n \)-divided prime ideal of \( R \). Note that if \( n = 1 \), then a divided ring in the sense of Dobbs-Badawi is a 1-divided ring.
Corollary

Assume that an integral domain $R$ is an $n$-PVD for some positive integer $n \geq 1$. Then $R$ is an $n$-divided domain and the set of all prime ideals of $R$ are linearly ordered by inclusion.

Definition

Let $n \geq 1$ be a positive integer and $S$ be a subset of an integral domain $R$ with quotient field $K$. Then $E_n(S) = \{x \in K \mid x^n \not\in S\}$ and $A_n(S) = \{x^n \mid x^n \in S, x \in K\}$.

Theorem

Let $n \geq 1$, $R$ be an integral domain, and $P$ be a prime ideal of $R$. Then $P$ is a $n$-powerful semiprimary ideal of $R$ if and only if $x^{-n}d \in P$ for every $x \in E_n(P)$ and every $d \in A_n(P)$. 
Corollary

Let $n \geq 1$ and $R$ be a quasilocal domain with maximal ideal $M$. The following statements are equivalent.

1. $R$ is an $n$-PVD.
2. $x^{-n}d \in R$ for every $x \in E_n(R)$ and every $d \in A_n(M)$.
3. $x^{-n}d \in M$ for every $x \in E_n(M)$ and every $d \in A_n(M)$.

Theorem

Let $n \geq 1$ and $R$ be a root closed integral domain with quotient field $K$. Then $R$ is a PVD if and only if $R$ is an $n$-PVD.

We recall from Anderson-Zafrulla that an integral domain $R$ with quotient filed $K$ is called an *almost valuation domain* if for every nonzero $x \in K$, there is an integer $n \geq 1$ ($n$ depends on $x$) such that $x^n \in R$ or $x^{-n} \in R$. We have the following definition.
Definition
Let \( n \geq 1 \) be a positive integer and \( R \) be an integral domain with quotient field \( K \) is called an \( n \)-valuation domain (\( n \)-VD) if for every nonzero \( x \in K \), we have \( x^n \in R \) or \( x^{-n} \in R \). It is clear that an \( n \)-valuation domain is an almost valuation domain.

Theorem
Let \( R \) be an \( n \)-PVD for some positive integer \( n \geq 1 \) with maximal ideal \( M \). Suppose that \( V \) is an overring of \( R \) such that \( \frac{1}{s} \in V \) for some \( s \in M \). Then \( V \) is an \( n \)-VD (and hence \( V \) is an almost valuation domain).

Theorem
Let \( R \) be an \( n \)-PVD for some positive integer \( n \geq 1 \) with maximal ideal \( M \). Suppose that \( P \) is a prime ideal of \( R \) such that \( P \neq M \). Then \( R_P \) is an \( n \)-VD (and hence \( R_P \) is an almost valuation domain). Furthermore, \( x^n \in R \) for every \( x \in P_P \), and hence \( P_P \subset R \).
Theorem

Suppose that an integral domain $R$ with quotient field $K$ admits a principal prime ideal $P$ of $R$ that is an $n$-divided ideal of $R$ for some positive integer $n \geq 1$, then $P$ is a maximal ideal of $R$. In particular, if $P$ is an $n$-powerful semiprimary ideal of $R$ for some positive integer $n \geq 1$, then $P$ is a maximal ideal of $R$ and $R$ is an $n$-VD.

Theorem

Let $n \geq 1$, $R$ be an integral domain with quotient field $K$ and $P$ be a prime ideal of $R$. Assume that $P$ is an $n$-powerful semiprimary ideal of $R$. Then $P$ is an $mn$-powerful semiprimary ideal of $R$ for every integer $m \geq 1$. Furthermore, if $x^{mn} \in P$ for some integer $m \geq 1$ and $x \in K$, then $x^n \in P$. In particular, if $R$ is an $n$-PVD, then $R$ is an $mn$-PVD for every integer $m \geq 1$. 
Theorem

Let \( n \geq 1 \) be an integer and \( R \) be an \( n \)-PVD with maximal ideal \( M \) and with quotient field \( K \). Assume that \( B \) is an overring of \( R \) that is integral over \( R \). Then \( B \) is an \( n \)-PVD with maximal ideal \( \sqrt{MB} = \{ x \in B \mid x^n \in M \} \).

Theorem

Let \( n \geq 1 \) be an integer and \( R \) be an \( n \)-PVD with maximal ideal \( M \) and with quotient field \( K \). Then \( \overline{R} \) is a PVD (1-PVD) with maximal ideal \( N = \{ x \in K \mid x^n \in M \} \).

Theorem

Let \( n \geq 1 \) be an integer and \( R \) be a quasilocal domain with maximal ideal \( M \) and with quotient field \( K \). Then \( R \) is an \( n \)-PVD if and only if \( \overline{R} \) is a PVD with maximal ideal \( N = \{ x \in K \mid x^n \in M \} \).
Corollary

Let \( n \geq 1 \) be an integer and \( R \) be a quasilocal domain with maximal ideal \( M \) and with quotient field \( K \). The following statements are equivalent.

1. \( R \) is an \( n \)-PVD.
2. \( \overline{R} \) is a PVD with maximal ideal \( N = \{ x \in K \mid x^n \in M \} \).
3. \( N = \{ x \in K \mid x^n \in M \} \) is a maximal ideal of \( \overline{R} \) such that \( (N : N) \) is a valuation domain with maximal ideal \( N \).

Theorem

Let \( n \geq 1 \) be an integer and \( R \) be an \( n \)-PVD with maximal ideal \( M \) and with quotient field \( K \). Then every overring of \( R \) is an \( n \)-PVD if and only if \( \overline{R} \) is a valuation domain.