Factorization in complement-finite ideals of free monoids

Nicholas R. Baeth
(Joint work with Matthew Enlow)

March 23, 2019

Franklin & Marshall College

Multiplication in Numerical Semigroups

Let $S = \langle n_1, \ldots, n_t \rangle = \{a_1 n_1 + \cdots + a_t n_t \colon a_i \in \mathbb{N}_0\}$ be a numerical semigroup, a complement-finite additive subsemigroup of $(\mathbb{N}_0, +)$.

- $S\backslash\{0\}$ is a cancellative multiplicative submonoid of $\mathbb{N}$.
- $|\mathbb{N}\backslash S| < \infty$
- For all $s \in S$ and all $n \in \mathbb{N}$, $ns \in S$.

$S\backslash\{0\}$ is a complement-finite ideal of the free multiplicative submonoid $\mathbb{N}$.

Seemingly nice subsemigroups

Let S be a complement-finite ideal of F; that is, a (multiplicative) submonoid of a free (reduced) monoid F such that:

- $|F \backslash S| < \infty$
- $fs \in S \; \forall s \in S \backslash \{1\}$ and $f \in F$

Examples

1. $S \backslash \{0\} \subseteq \mathbb{N}$ *where* S *is a numerical semigroup*
2. $S = \mathbb{N} \backslash \{p, p^2, \ldots, p^k\} \subseteq \mathbb{N}$ *with* p *prime and* $k \geq 1$.
3. $S = \mathbb{N} \backslash \{p^a q^b : p, q \in \mathbb{P}, (a, b) \in A \subseteq \mathbb{N}_0 \times \mathbb{N}_0\}$.
4. *Generalizations of 3.*

Examples

$$S = \langle 3, 4, 5 \rangle = \{3, 4, 5, \ldots\}$$

Examples

$S = \langle 3, 4, 5 \rangle = \{3, 4, 5, \ldots\}$

- $\mathcal{A}(S) = \{p \in \mathbb{P}_{\geq 3}\} \cup \{2p \colon p \in \mathbb{P}_{\geq 3}\} \cup \{4, 8\}$

Examples

$S = \langle 3, 4, 5 \rangle = \{3, 4, 5, \ldots\}$

- $\mathcal{A}(S) = \{p \in \mathbb{P}_{\geq 3}\} \cup \{2p : p \in \mathbb{P}_{\geq 3}\} \cup \{4, 8\}$
- $\alpha = 4^a 8^b \implies$

$$\mathsf{L}(\alpha) = \left[\left\lceil \frac{\mathsf{v}_2(\alpha)}{3} \right\rceil, \left\lfloor \frac{\mathsf{v}_2(\alpha)}{2} \right\rfloor\right].$$

Examples

$S = \langle 3, 4, 5 \rangle = \{3, 4, 5, \ldots\}$

- $\mathcal{A}(S) = \{p \in \mathbb{P}_{\geq 3}\} \cup \{2p : p \in \mathbb{P}_{\geq 3}\} \cup \{4, 8\}$
- $\alpha = 4^a 8^b \implies$
$$\mathsf{L}(\alpha) = \left[\left\lceil \frac{\mathsf{v}_2(\alpha)}{3} \right\rceil, \left\lfloor \frac{\mathsf{v}_2(\alpha)}{2} \right\rfloor\right].$$
- $\alpha = p_1 \cdots p_s (2q_1) \cdots (2q_t) 4^a 8^b \implies$
$$\mathsf{L}(\alpha) = \left[(s+t), (s+t) + \left\lfloor \frac{\mathsf{v}_2(\alpha)}{2} \right\rfloor\right] \text{ if } s >> 0.$$

Examples

$S = \langle 3, 4, 5 \rangle = \{3, 4, 5, \ldots\}$

- $\mathcal{A}(S) = \{p \in \mathbb{P}_{\geq 3}\} \cup \{2p \colon p \in \mathbb{P}_{\geq 3}\} \cup \{4, 8\}$
- $\alpha = 4^a 8^b \implies$
$$\mathsf{L}(\alpha) = \left[\left\lceil \frac{\mathsf{v}_2(\alpha)}{3} \right\rceil, \left\lfloor \frac{\mathsf{v}_2(\alpha)}{2} \right\rfloor\right].$$
- $\alpha = p_1 \cdots p_s (2q_1) \cdots (2q_t) 4^a 8^b \implies$
$$\mathsf{L}(\alpha) = \left[(s+t), (s+t) + \left\lfloor \frac{\mathsf{v}_2(\alpha)}{2} \right\rfloor\right] \text{ if } s >> 0.$$
- $\rho(S) = 3/2.$

Examples

$S = \mathbb{N} \backslash \{p, p^2, \ldots, p^{k-1}\} \subseteq \mathbb{N}$ where p is prime in $\mathbb{N}$ and $k \geq 2$.

Examples

$S = \mathbb{N}\backslash\{p, p^2, \ldots, p^{k-1}\} \subseteq \mathbb{N}$ where p is prime in $\mathbb{N}$ and $k \geq 2$.

- $\mathcal{A}(S) = \{p^i q \colon q \in \mathbb{P}\backslash\{p\}, 0 \leq i \leq k\} \cup \{p^k, \ldots, p^{2k-1}\}$

Examples

$S = \mathbb{N} \setminus \{p, p^2, \dots, p^{k-1}\} \subseteq \mathbb{N}$ where p is prime in $\mathbb{N}$ and $k \geq 2$.

- $\mathcal{A}(S) = \{p^i q \colon q \in \mathbb{P} \setminus \{p\}, 0 \leq i \leq k\} \cup \{p^k, \dots, p^{2k-1}\}$
- $\alpha = p^a \implies \mathsf{L}(\alpha) = \left[\!\left[\left\lceil \frac{a}{2k-1} \right\rceil, \left\lfloor \frac{a}{k} \right\rfloor \right]\!\right].$

Examples

$S = \mathbb{N} \backslash \{p, p^2, \ldots, p^{k-1}\} \subseteq \mathbb{N}$ where p is prime in $\mathbb{N}$ and $k \geq 2$.

- $\mathcal{A}(S) = \{p^i q \colon q \in \mathbb{P} \backslash \{p\}, 0 \leq i \leq k\} \cup \{p^k, \ldots, p^{2k-1}\}$
- $\alpha = p^a \Longrightarrow \mathsf{L}(\alpha) = \left[\left\lceil \frac{a}{2k-1} \right\rceil, \left\lfloor \frac{a}{k} \right\rfloor\right]$.
- $\alpha = q_1 \cdots q_s p^a \Longrightarrow \mathsf{L}(\alpha) = \left[s, s + \left\lfloor \frac{a}{k} \right\rfloor\right]$ if $s >> 0$.

Examples

$S = \mathbb{N}\backslash\{p, p^2, \ldots, p^{k-1}\} \subseteq \mathbb{N}$ where p is prime in $\mathbb{N}$ and $k \geq 2$.

- $\mathcal{A}(S) = \{p^i q \colon q \in \mathbb{P}\backslash\{p\}, 0 \leq i \leq k\} \cup \{p^k, \ldots, p^{2k-1}\}$
- $\alpha = p^a \implies \mathsf{L}(\alpha) = [\lceil \frac{a}{2k-1} \rceil, \lfloor \frac{a}{k} \rfloor]$.
- $\alpha = q_1 \cdots q_s p^a \implies \mathsf{L}(\alpha) = [s, s + \lfloor \frac{a}{k} \rfloor]$ if $s >> 0$.
- $\rho(S) = \frac{2k-1}{k}$.

Examples

$S = \langle 6, 2^k, 3^k \rangle$

Examples

$S = \langle 6, 2^k, 3^k \rangle$

- Irreducibles that divide 6^{2k-1} are: $6, 2^k, \ldots, 2^{2k-1}$, $3^k, \ldots, 3^{2k-1}$, $6 \cdot 2^i$, $6 \cdot 3^i$ with $1 \leq i \leq k-1$

Examples

$S = \langle 6, 2^k, 3^k \rangle$

- Irreducibles that divide 6^{2k-1} are: $6, 2^k, \ldots, 2^{2k-1}$, $3^k, \ldots, 3^{2k-1}$, $6 \cdot 2^i$, $6 \cdot 3^i$ with $1 \leq i \leq k-1$
- $6^{2k-1} = (2^{2k-1})(3^{2k-1})$ and so $\rho(S) \geq \rho(6^{2k-1}) = k - \frac{1}{2}$.

Examples

$S = \langle 6, 2^k, 3^k \rangle$

- Irreducibles that divide 6^{2k-1} are: $6, 2^k, \ldots, 2^{2k-1}$, $3^k, \ldots, 3^{2k-1}$, $6 \cdot 2^i$, $6 \cdot 3^i$ with $1 \leq i \leq k-1$
- $6^{2k-1} = (2^{2k-1})(3^{2k-1})$ and so $\rho(S) \geq \rho(6^{2k-1}) = k - \frac{1}{2}$.
- $L(6^{2k-1}) = [2, 2k-1]$

Irreducible Elements

Let S be a complement-finite ideal of a free (reduced) monoid $F = \mathcal{F}(P)$.

The irreducible elements of S are those with the following forms:

1. $p \in P \cap S$
2. px with $p \in P \cap S$ and $x \in F \setminus S$
3. $q_1^{r_1} \cdots q_t^{r_t}$ with $q_1, \ldots q_t \in P \setminus S$ and $(r_1, \ldots, r_t)$ almost minimal

Moreover, no irreducible element is prime in S.

For each $s \in S$, the combined number of irreducibles of types (1) and (2) is independent of the factorization.

Factorizations

S a complement-finite ideal of $F = \mathcal{F}(P)$

$P \backslash S = \{p_1, \ldots, p_t\}$ with $k_i = \min\{k \colon p_i^k \in S\}$

- $n \geq k_i \Longrightarrow \mathsf{L}_S(p_i^n) = L_T(n)$ where $T = \langle k_i, \ldots, 2k_i - 1\rangle$.
- $n_i \geq k_i \,\forall\, i \Longrightarrow \mathsf{L}(p_1^{n_1} \cdots p_t^{n_t}) \supseteq \sum_{i=1}^{t} \mathsf{L}_S(p_i^{n_i})$
- $\rho(S) \leq \frac{M}{m}$ where
  $M = \max\{n_1 + \cdots + n_t \colon p_1^{n_1} \cdots p_t^{n_t} \in \mathcal{A}(S)\}$ and
  $m = \min\{n_1 + \cdots + n_t \colon p_1^{n_1} \cdots p_t^{n_t} \in \mathcal{A}(S)\}$
- $N \in \bigcap_{i=1}^{t} [k_i, 2k_i - 1]$ and $\alpha = p_1^{a_1} \cdots p_t^{a_t} \in \mathcal{A}(S)$
  $\Longrightarrow \alpha^N = (p_1^N)^{a_1} \cdots (p_t^N)^{a_t}$ and $\rho(\alpha) \geq \frac{N}{a_1 + \cdots + a_n}$.

C-monoid structure

S a complement-finite ideal of a (reduced) free monoid F.

1. Then S is not a Krull monoid.
   [It's not completely integrally closed.]

2. S is a C-monoid. Moreover, the class semigroup $\mathcal{C}^*(S, F)$ has exactly two idempotent elements: $\{1\}$ and S.

Recall that $\mathcal{C}^*(S, F) = \{[x] \colon x \in F\}$ with $[x] = [y]$ whenever $xa \in S \Leftrightarrow ya \in S$, and S is a C-monoid when $|\mathcal{C}^*(S, F)| < \infty$.

A Transfer Homomorphism

Let S be an complement-finite ideal of a free monoid F and let $C = \mathcal{C}^*(S, F) = \{e, c_1, \ldots, c_n, h\}$ denote its class semigroup, where $e = \{1\}$ and $h = S \backslash \{1\}$ are the two idempotent elements.

Let

$$\mathcal{B}^*(S) = \{ \underbrace{c_1^{v_1} \cdots c_n^{v_n}}_{\text{formal product}} : \underbrace{\prod c_i^{v_i}}_{\text{actual product}} = h\} \subseteq \mathcal{F}(\{c_1, \ldots, c_n\}).$$

The natural projection from S to $\mathcal{B}^*(S)$ is a transfer homomorphism.