Polynomials with half-factorial coefficients

Mark Batell
Fairfield University

March 23, 2019

Definition. An integral domain R is **half-factorial** if R is atomic and $m = n$ whenever $\alpha_1\alpha_2\cdots\alpha_n = \beta_1\beta_2\cdots\beta_m$ and the α's and β's are irreducible elements (atoms) of R.

Gauss's Lemma

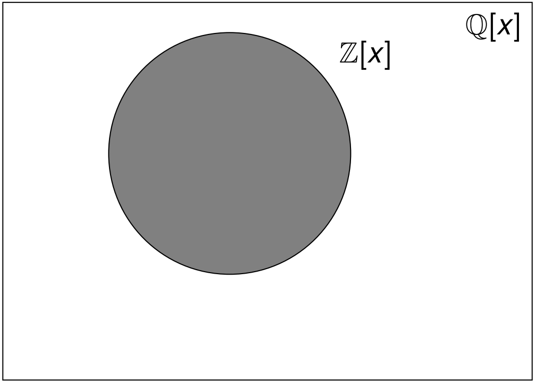

Figure: Venn diagram illustrating the containment of polynomial rings $\mathbb{Z}[x] \subseteq \mathbb{Q}[x]$. Gauss used the fact that $\mathbb{Q}[x]$ is factorial to prove that $\mathbb{Z}[x]$ is also factorial.

Gauss' Lemma Part I The product of two primitive polynomials over $\mathbb{Z}$ is primitive.

Gauss' Lemma Part II Every irreducible polynomial $f \in \mathbb{Z}[x]$ of degree $\geqslant 1$ is irreducible in $\mathbb{Q}[x]$.

Theorem (Gauss) The polynomial ring $\mathbb{Z}[x]$ is factorial.

Question: Under what conditions is a polynomial ring $R[x]$ half-factorial?

Theorem (Zaks) Let R be a Krull domain with class group G. Then the polynomial ring $R[x]$ is half-factorial if and only if $|G| \leqslant 2$.

Sufficient Conditions

Theorem. Consider the following conditions on a domain R:

a) R is integrally closed
b) If $f, g \in R[x]$ and fg is primitive, then f is superprimitive or g is superprimitive
c) If f, g are primitive polynomials over R and $a \in R$ is a constant factor of the product fg, then $\ell(a) \leqslant 1$
d) If I is a primitive ideal and $\frac{r}{s} \in I^{-1}$, then there is a common divisor g of r and s such that $\ell(\frac{s}{g}) \leqslant 1$.

If R is an atomic domain satisfying all of the above, then $R[x]$ is an HFD.

Euclid's Lemma: In a factorial domain, if

$$ab = cd$$

where a and c are relatively prime, then a divides d.

A Generalization: Let us say that R has the **Z-property** if whenever

$$abc = de$$

(where the elements are all nonzero nonunits), then either ab and d are not relatively prime or ab and e are not relatively prime.

Example. The Krull domain $R := F[x, y, zx, zy]$ does not have the Z-property:

$$(x)(x)(zy)(zy) = (zx)(zx)(y)(y)$$

Integral Closures

Theorem (Mori, Nagata) The integral closure of a Noetherian domain is a Krull domain.

Theorem (Barucci) The complete integral closure of an integrally closed Mori domain is a Krull domain.

Example. The Mori domain $R := \mathbb{Z} + x6\mathbb{Z}[x]$ does not have the Z-property:

$$(2)(3)(6x^2) = (6x)(6x)$$

Theorem: If the polynomial ring $R[x]$ is half-factorial, then R has the Z-property.

Theorem: Let R be atomic. If R has the Z-property, then R is half-factorial.

Characterization of half-factorial polynomial rings

Theorem: Assume R is a domain in which every v-finite v-ideal is v-generated by two elements. Then $R[x]$ is an HFD if and only if each of the following conditions is satisfied:

(1) R is integrally closed,

(2) R has the Z-property, and

(3) $(AB)^{-1} = \{uv \mid u \in A^{-1}, v \in B^{-1}\}$ if $A, B \subseteq R$ are finitely generated ideals whose product AB is primitive.

The two-generator problem for ideals

Theorem (Matlis). Let R be a Noetherian integral domain. Then every ideal is generated by two elements if and only if

(1) R is a one-dimensional Gorenstein ring, and

(2) M is projective or M^{-1} is a one-dimensional Gorenstein ring for each maximal ideal M of R.

Open Problem: Generalize this result to v-ideals.

Example. In $F[[x^3, x^5]]$, consider the ideal

$$I = M^2 = (x^6, x^8, x^{10})$$

Is it v-generated by two elements? The following calculations suggest otherwise:

$$1/x^3 \in (x^6, x^8)^{-1}$$

$$1/x \in (x^6, x^{10})^{-1}$$

$$1/x^2 \in (x^8, x^{10})^{-1}$$

$$1/x - 1/x^3 \in (x^6, x^8 + x^{10})^{-1}$$

Theorem. Let R be a Krull domain. Then R has the Z-property if and only if R is half-factorial and for each primitive ideal $I \subseteq R$ there exists an irreducible element α such that $\alpha \in I_v$.

An Application. The Krull domain $R := \mathbb{Q}[x, y, zx, zy]$ is not half-factorial:

$$\left(x^2 + y^2\right)\left(x^2 + z^2x^2\right) = x^2(x^2 + y^2 + z^2x^2 + z^2y^2)$$

Lemma. Assume R is a Krull domain with nontorsion class group. Then there exists a nontorsion prime P and an irreducible element $x \in P$ such that x^2 factors uniquely.

Theorem. Let R be a Krull domain. If R has the Z-property, then its class group is torsion.