On Numerical Semigroups

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March 23, 2019

Basic notions

- Gaps, non-gaps, genus, gapsets, Frobenius number, conductor
- Generators

Counting by genus

- Conjecture
- Dyck paths and Catalan bounds
- Semigroup tree and Fibonacci bounds
- Ordinarization transform and ordinarization tree
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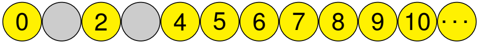
$$G \text{ gapset} \Longleftrightarrow \mathbb{N}_0 \setminus G \text{ numerical semigroup.}$$

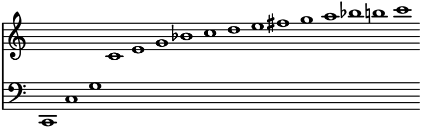
Cash point

The amounts of money one can obtain from a cash point (divided by 10)

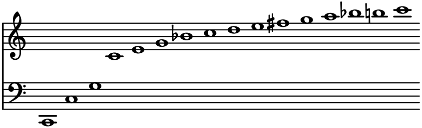
Illustration: Agnès Capella Sala



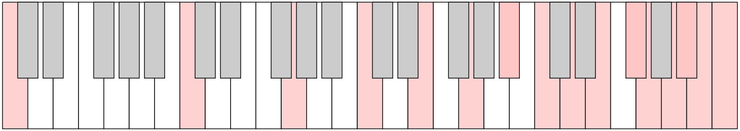
Harmonics



Harmonics: 12-semitone count



Divide the octave into 12 equal semitones.



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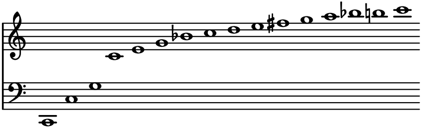

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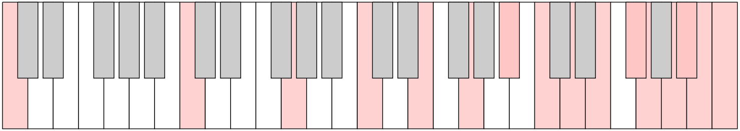

What semitone interval corresponds to each harmonic?

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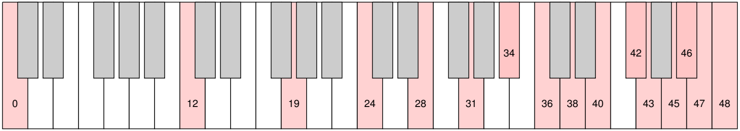

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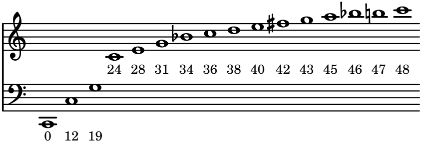

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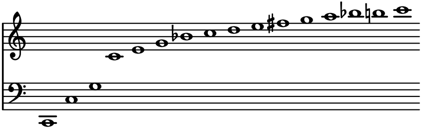

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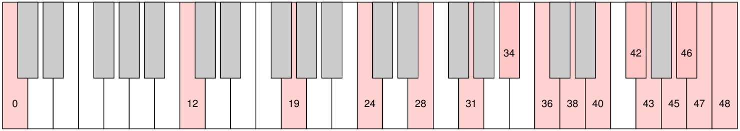

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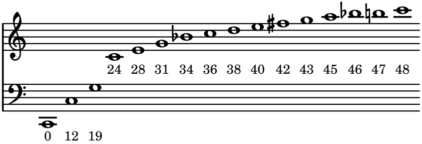

$H = \{0, 12, 19, 24, 28, 31, 34, 36, 38, 40, 42, 43, 45, 46, 47, 48, 49, 50, \rightarrow\}.$

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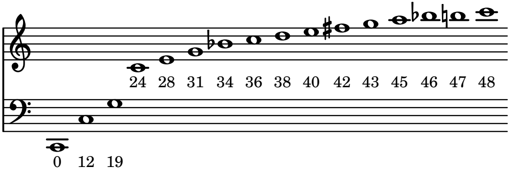
Frobenius number $:=$ the largest gap F
conductor $:= c = F + 1$

The Well-tempered semigroup



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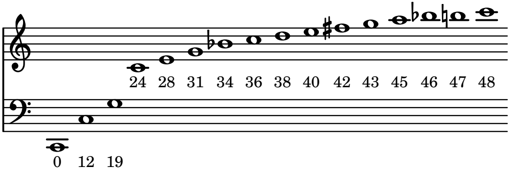

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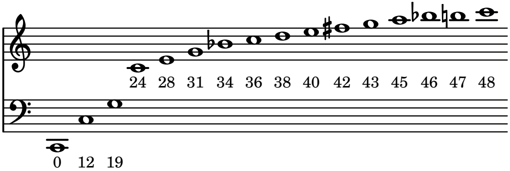

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- $g = 33$
- $F = 44$
- $c = 45$

Generators

The generators of a numerical semigroup are those non-gaps which can not be obtained as a sum of two smaller non-gaps.

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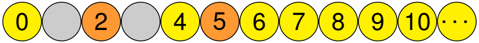

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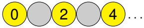

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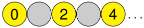

- $n_3 = 4$
- $n_4 = 7$
- $n_5 = 12$
- $n_6 = 23$
- $n_7 = 39$
- $n_8 = 67$

$\vdots$

Counting semigroups by genus

Conjecture

[B-A 2008]

1. $n_g \geqslant n_{g-1} + n_{g-2}$
2. - $\lim_{g\to\infty} \frac{n_{g-1}+n_{g-2}}{n_g} = 1$
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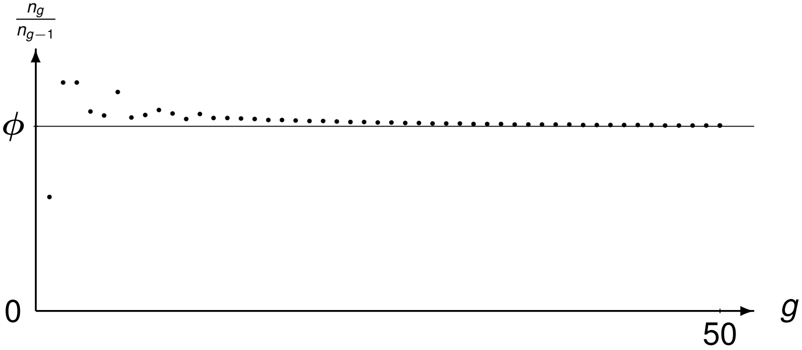
Weaker unsolved conjecture

[B-A 2007]

$$n_g \leqslant n_{g+1}$$

Counting semigroups by genus

Behavior of $\frac{n_g}{n_{g-1}}$



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What is known

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- $\lim_{g\to\infty}\frac{n_g}{n_{g-1}}=\phi$
  Alex Zhai (2013) with important contributions of Nathan Kaplan and Yufei Zhao.

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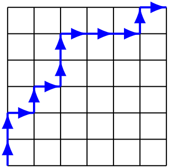

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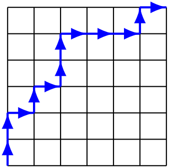

The number of Dyck paths of order n is given by the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Dyck paths

Definition

The square diagram of a numerical semigroup is the path

$$e(i) = \begin{cases} \rightarrow & \text{if } i \in \Lambda, \\ \uparrow & \text{if } i \notin \Lambda, \end{cases} \qquad \text{for } 1 \leqslant i \leqslant 2g.$$

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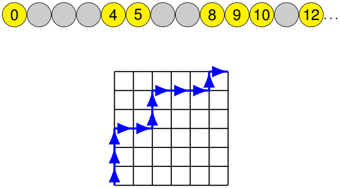

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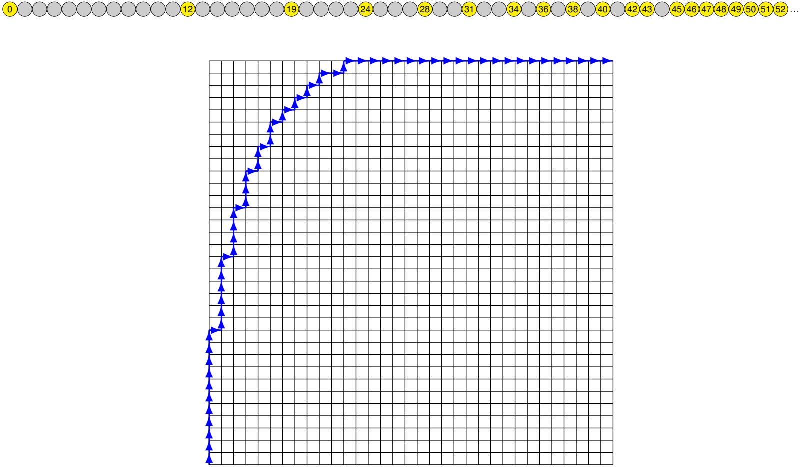

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Lemma

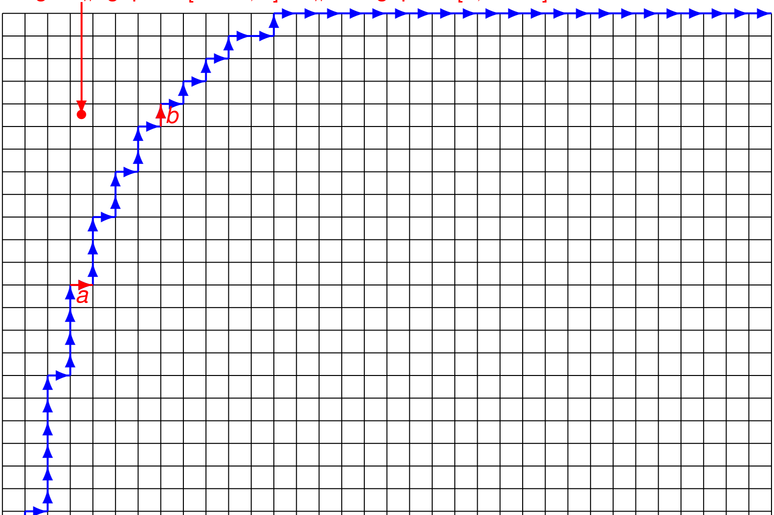
The weight of a numerical semigroup $\left(\sum_{l_i : i\text{th gap}}(l_i - i)\right)$ *is the area over the path of the numerical semigroup in the square* $[0, g]^2$.

Dyck paths

Not all Dyck paths correspond to numerical semigroups.

Let us parallel the results in Nathan's talk. Use the *augmented Dyck path* (starting the path from 0 instead of 1) and compute hook lengths.

hook length=$\#$ gaps in $[a+1, b]$ + $\#$ nongaps in $[a, b-1] - 1 = b - a$



Dyck paths

Consequently,

$$H(D) = \{b - a : b \text{ a gap }, a \text{ a nongap}\},$$

while the hook lengths in the first column are

$$h(D) = \{b : b \text{ a gap }\}.$$

Now, by the gapset definition, an (augmented) Dyck path corresponds to a numerical semigroup if and only $H(D) \subseteq h(D)$ [Constantin, Houston-Edwards, Kaplan]

Tree $\mathcal{T}$ of numerical semigroups

From genus g to genus $g-1$

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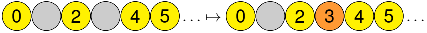

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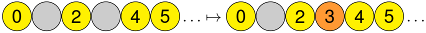
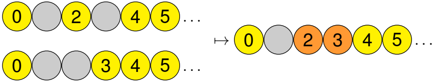


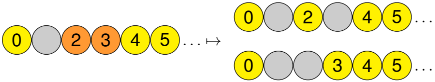
Not injective



Tree $\mathcal{T}$ of numerical semigroups

From genus $g-1$ to genus g

All semigroups giving Λ when adjoining to them their Frobenius number can be obtained from Λ by taking out one by one all generators of Λ larger than its Frobenius number.



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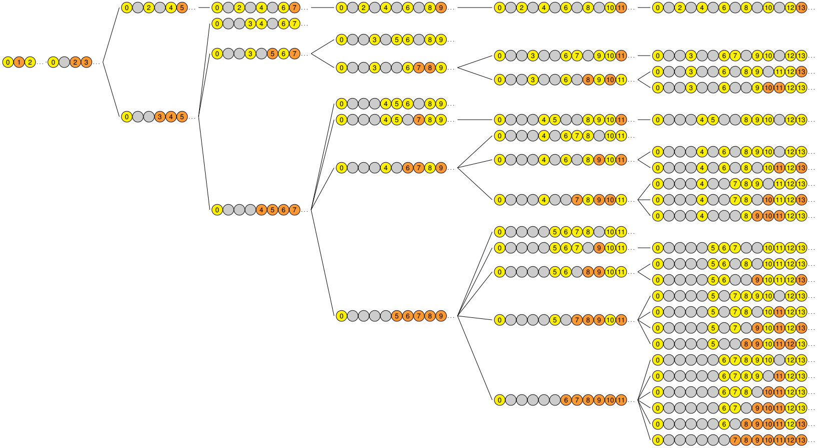

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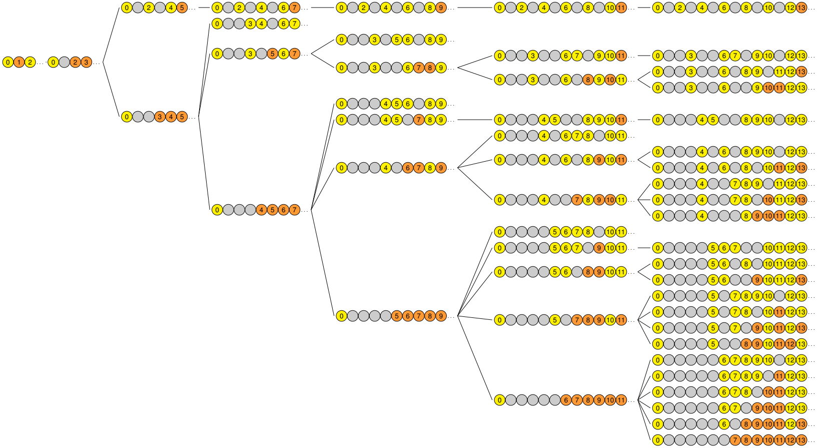

The parent of a semigroup Λ is Λ together with its Frobenius number.

The descendants of a semigroup are obtained taking away one by one all generators larger than its Frobenius number.

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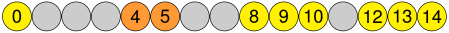

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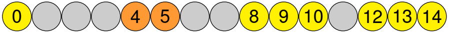

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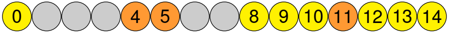

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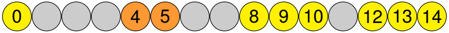

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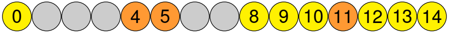

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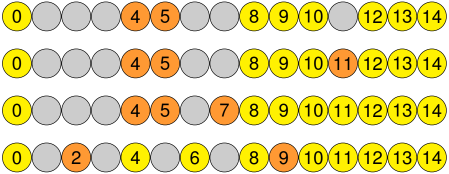

… has 0 descendants

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… has ∞ descendants

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Theorem (B-A, Bulygin, 2009)

Let $d = \gcd(\lambda_1, \dots, \lambda_{c-g-1})$. *Then,*

1. Λ *has* ∞ *descendants* $\iff d \neq 1$.
2. *If* $d \neq 1$ *then* Λ *lies in infinitely many infinite chains if and only if* d *is not prime.*

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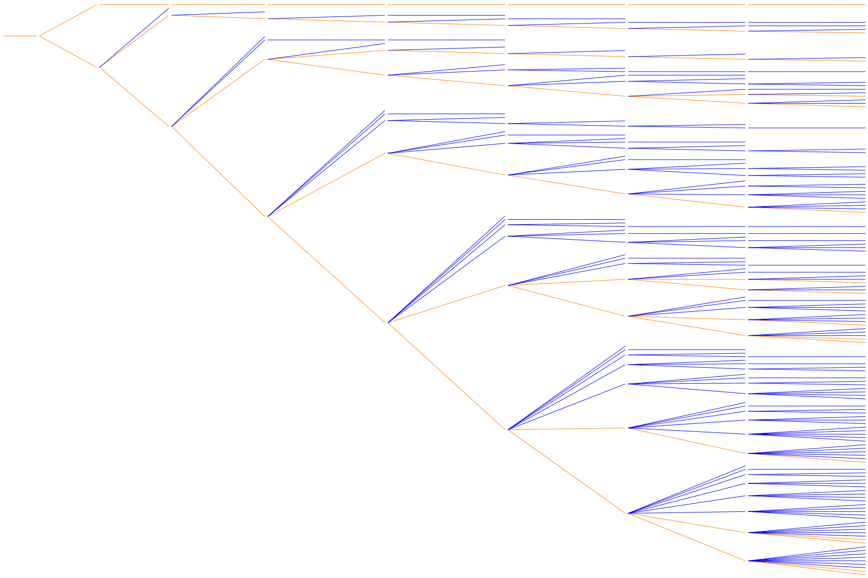
Computation shows that most numerical semigroups have a finite number of descendants.

Tree $\mathcal{T}$ of numerical semigroups

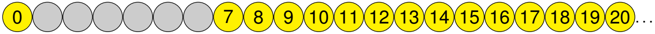


Tree $\mathcal{T}$ of numerical semigroups

We want to analyze the number of descendants of a node in terms of the number of descendants of its parent.

Tree $\mathcal{T}$ of numerical semigroups

A numerical semigroup is ordinary if all its gaps are consecutive.



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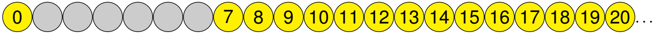

Descendants of ordinary semigroups

Lemma

If the node of an ordinary semigroup has k descendants, then its descendants have $0, 1, \ldots, k-3, k-1, k+1$ descendants, respectively.

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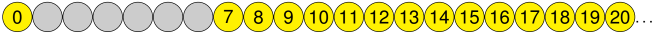

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Descendants of nonordinary semigroups

Lemma

If a non-ordinary node in the semigroup tree has k descendants, then its descendants have

- *at least $0, \ldots, k-1$ descendants, respectively,*
- *at most $1, \ldots, k$ descendants, respectively.*

Bounds using descending rules

Lemma

For $g \geqslant 3$,

$$2F_g \leqslant n_g \leqslant 1 + 3 \cdot 2^{g-3}.$$

Basic notions

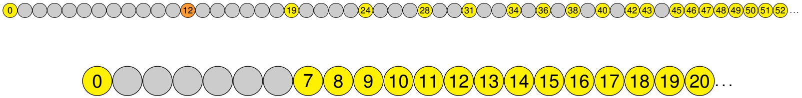
- Gaps, non-gaps, genus, gapsets, Frobenius number, conductor
- Generators

Counting by genus

- Conjecture
- Dyck paths and Catalan bounds
- Semigroup tree and Fibonacci bounds
- Ordinarization transform and ordinarization tree
- Quasi-ordinarization transform and quasi-ordinarization forest

Ordinary numerical semigroups

The multiplicity of a numerical semigroup is its smallest non-zero nongap.



Ordinarization of semigroups

Ordinarization transform of a semigroup:

- Remove the multiplicity
- Add the Frobenius number

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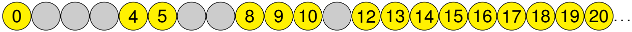

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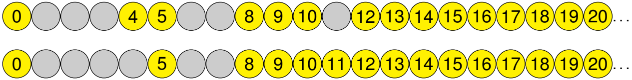

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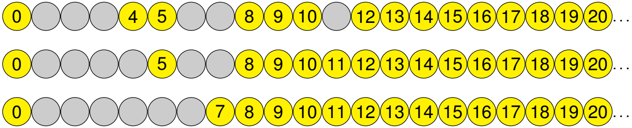

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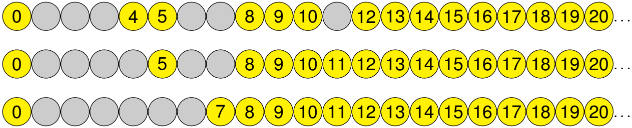

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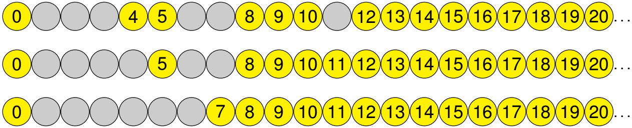

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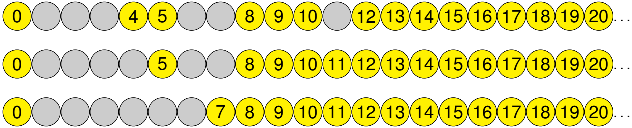

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- Repeating several times (:= ordinarization number) we obtain an ordinary semigroup.

Tree $\mathcal{T}_g$ of numerical semigroups of genus g

The tree $\mathcal{T}_g$

Define a graph with

- **nodes** corresponding to semigroups of genus g
- **edges** connecting each semigroup to its ordinarization transform

$$o(\Lambda) - \Lambda$$

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Contrary to $\mathcal{T}$, $\mathcal{T}_g$ has only a finite number of nodes (indeed, n_g).

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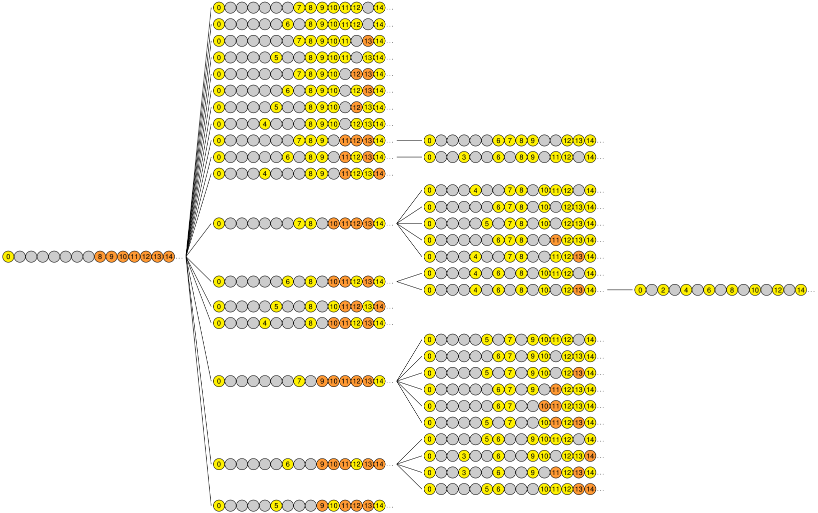

Conjecture

$n_{g,r}$: number of semigroups of genus g and ordinarization number r.

Conjecture

- $n_{g,r} \leqslant n_{g+1,r}$
- Equivalently, the number of semigroups in $\mathcal{T}_g$ at a given depth is at most the number of semigroups in $\mathcal{T}_{g+1}$ at the same depth.

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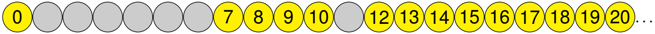
This result is proved for the lowest and largest depths.

Quasi-ordinary numerical semigroups

A non-ordinary semigroup Λ is a quasi-ordinary semigroup if $\Lambda \cup F$ is ordinary.



Quasi-ordinarization of semigroups

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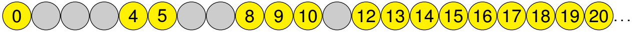

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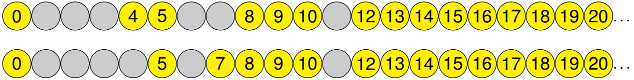

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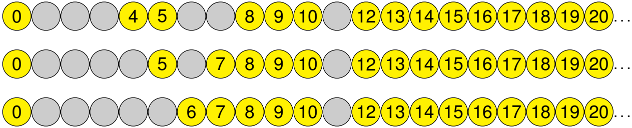

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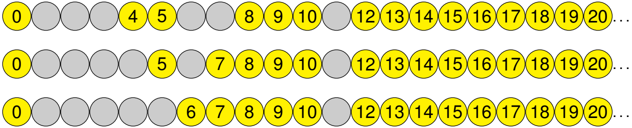

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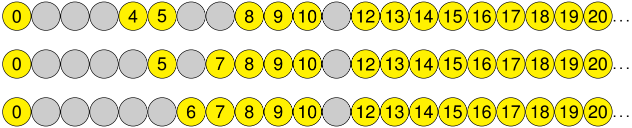

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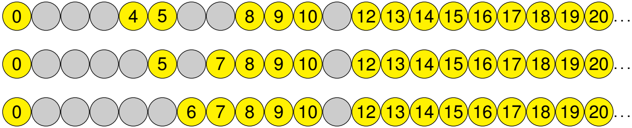

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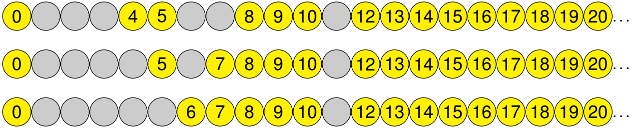

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Quasi-ordinarization transform of an ordinary semigroup is defined to be itself.

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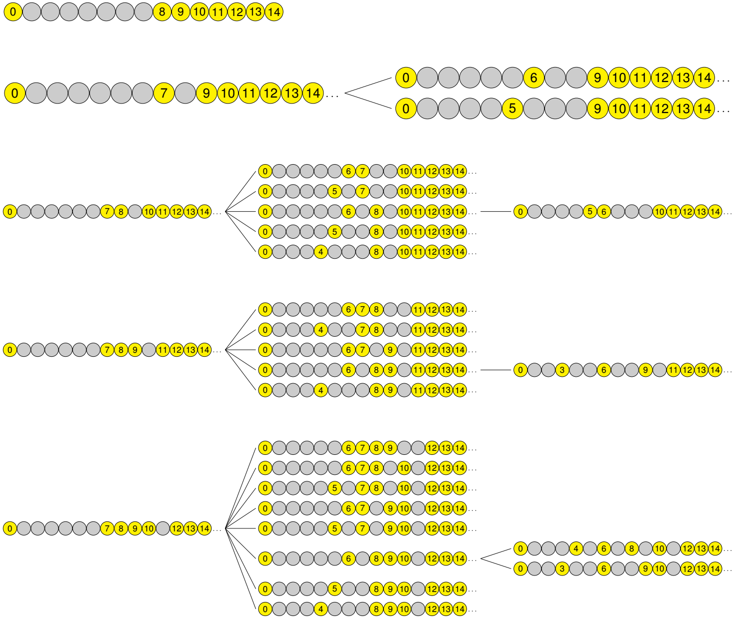
Contrary to $\mathcal{T}_g$, $\mathcal{F}_g$ is a forest.

Forest $\mathcal{F}_g$ of numerical semigroups of genus g



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This conjecture would prove $n_g \leqslant n_{g+1}$.

Recommended...

Recommended reference:
Nathan Kaplan. Counting numerical semigroups, Amer. Math. Monthly 124: 862-875, 2017.

Recommended website:
Combinatorial Object Server++ Maintained by Torsten Mütze.

Numerical semigroups arise in

- Algebraic geometry
  (as Weierstrass semigroups, see general references)
- Coding theory
  (see for example Numerical Semigroups and Codes)
- Privacy models
  (see Klara Stokes' PhD thesis and later works)
- Music theory
  (Tempered monoids, the golden fractal monoid, and the well-tempered harn