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Sets of Arithmetical Invariants in Transfer Krull Monoids

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Integral Domains and Monoids

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Outline

Sets of arithmetical invariants

Transfer Krull monoids

Main Results

Open Problems

Sets of lengths in monoids

Monoid H: multiplicatively written, cancellative semigroup, with unit element.

Let $a \in H$:

- If $a = u_1 \cdot \ldots \cdot u_k$ where $u_1, \ldots, u_k \in \mathcal{A}(H)$, then

  k is called the length of the factorization, and
- $\mathsf{L}_H(a) = \{k \mid a \text{ has a factorization of length } k\} \subset \mathbb{N}$ is the set of lengths of a.
- The system of all sets of lengths

$$\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$$

FACT 1. If H is commutative and v-noetherian, then all $\mathsf{L}(a)$ are finite and nonempty.

Set of Distances

- If $L = \{k_1, k_2, k_3, \ldots\} \subset \mathbb{N}$ with $k_1 < k_2 < k_3 < \ldots$, then

$$\Delta(L) = \{k_2 - k_1, k_3 - k_2, \ldots\}$$

  is the set of distances of L.

- $$\Delta(H) = \bigcup_{L \in \mathcal{L}(H)} \Delta(\mathrm{L}) \ \subset \ \mathbb{N}$$

  the set of distances of H.

- FACT 2. If $\Delta(H) \neq \emptyset$, then $\min \Delta(H) = \gcd \Delta(H)$.

Set of Elasticities

- For a finite set $L \subset \mathbb{N}$, let $\rho(L) = \max L / \min L$ be its elasticity.
- $\rho(H) = \sup\{\rho(L) : L \in \mathcal{L}(H)\} \in \mathbb{R}_{\geq 1} \cup \{\infty\}$ is the elasticity of H.
  - $\rho(H)$ is studied since the late 1980s.
  - Kainrath: Let R be a finitely generated domain. TFAE
    - $\rho(R) < \infty$
    - $\mathcal{C}(\overline{R})$ and $\overline{R}/R$ are finite and $\operatorname{spec}(\overline{R}) \to \operatorname{spec}(R)$ is injective.
- $\{\rho(L) \colon L \in \mathcal{L}(H)\} \subset \mathbb{Q}_{\geq 1}$ is the set of elasticities of H.
  - Baginski, Chapman et al.: 2006, 2007
  - García-Sánchez, Ponomarenko, ...
  - Recent work:
    - Gotti, O'Neill, Pelayo et al.: Numerical and Puiseux monoids.
    - Zhong: Structural results for the set of elasticities in locally finitely generated monoids.

Sets of lengths: Basic Facts

A monoid H is called half-factorial if one of the foll. equiv. holds:

(a) $|L| = 1$ for all $L \in \mathcal{L}(H)$.

(b) $\Delta(H) = \emptyset$.

(c) $\rho(H) = 1$.

FACT 3. If $a = u_1 \cdot \ldots \cdot u_k = v_1 \cdot \ldots \cdot v_\ell$ with $u_i, v_j \in \mathcal{A}(H)$, then

$$a^m = (u_1 \cdot \ldots \cdot u_k)^i (v_1 \cdot \ldots \cdot v_\ell)^{m-i} \quad \text{for all } i \in [0, m]$$

and hence

$$\mathsf{L}(a^m) \supset \{\ell m + i(k-\ell) \mid i \in [0, m]\} .$$

FACT 4. A monoid H is

- EITHER half-factorial

  OR
- For all $m \in \mathbb{N}$ there is $L \in \mathcal{L}(H)$ with $|L| > m$.

Distance between factorizations

Let H be commutative and $a \in H$. If $z, z' \in \mathsf{Z}(a)$ are two factorizations, say

$$z = u_1 \cdot \ldots \cdot u_n \, v_1 \cdot \ldots \cdot v_r \, , \quad z' = u_1 \cdot \ldots \cdot u_n \, w_1 \cdot \ldots \cdot w_s$$

where all u_i, v_j, w_k are atoms and

$$\{v_1, \ldots, v_r\} \cap \{w_1, \ldots, w_s\} = \emptyset \, .$$

then

$$\mathsf{d}(z, z') = \max\{r, s\} \quad \text{is the distance between } z \text{ and } z'.$$

FACT 5. If H is not factorial, then for every $N \in \mathbb{N}$ there exist $c \in H$ and factorizations $z, z' \in \mathsf{Z}(c)$ such that $|\mathsf{Z}(c)| > N$ and $\mathsf{d}(z, z') \geq 2N$.

Set of catenary degrees

- Let H be commutative and $a \in H$. Then $\mathsf{c}(a) \in \mathbb{N}_0$ is the smallest $N \in \mathbb{N}_0$ with the following property:

  For any $z, z' \in \mathsf{Z}(a)$, there exists a finite sequence

  $z = z_0, z_1, \ldots, z_k = z'$ concatenating z and z' in $\mathsf{Z}(a)$

  with $\mathsf{d}(z_{i-1}, z_i) \le N$ for all $i \in [1, k]$.
- $\mathsf{Ca}(H) = \{\mathsf{c}(a) \colon a \in H \text{ with } \mathsf{c}(a) > 0\}$ is the set of catenary degrees of H.
- $\mathsf{c}(H) = \sup \mathsf{Ca}(H)$ is the catenary degree of H.
- FACT 6.
  - $\mathsf{c}(a) = 0$ iff a has precisely one factorization.
  - H is factorial iff $\mathsf{c}(H) = 0$.
  - $\mathsf{c}(a) \le \max \mathsf{L}(a)$.
  - $2 + \max \Delta(H) \le \mathsf{c}(H)$.

Outline

Sets of arithmetical invariants

Transfer Krull monoids

Main Results

Open Problems

(Weak) Transfer Homomorphisms

G. + Halter-Koch, 1990s: Commutative cancellative setting
Baeth + Ponomarenko et al., 2011: Number theory of matrix sgr.
Baeth + Smertnig, 2014, 2015: Non-commutative setting
Fan + Tringali, 2018: Equimorphisms ...

Definition

A monoid homomorphism $\theta \colon H \to B$ is called a
a (weak) transfer homomorphism if it has the following properties:

(T 1) $B = B^\times \theta(H) B^\times$ and $\theta^{-1}(B^\times) = H^\times$.
is surjective up to units and

(WT 2) If $a \in H$ and $b_1, \ldots, b_n$ are atoms in B such that $\theta(a) = b_1 \cdot \ldots \cdot b_n$, then there exist atoms $a_1, \ldots, a_n \in H$ and a permutation $\sigma \in \mathfrak{S}_n$ such that $a = a_1 \cdot \ldots \cdot a_n$ and $\theta(a_i) = b_{\sigma(i)}$ for each $i \in [1, n]$.
allows to lift factorizations.

Weak Transfer Homomorphisms: Basic Properties

Let $\theta\colon H \to B$ be a weak transfer homomorphism between atomic monoids.

FACT 7. Let $a \in H$.

- $a \in H$ is an atom iff $\theta(a) \in B$ is an atom.
- $\mathsf{L}_H(a) = \mathsf{L}_B\big(\theta(a)\big)$.
- $\mathcal{L}(H) = \mathcal{L}(B)$, whence in particular
- $\Delta(H) = \Delta(B)$ and $\rho(H) = \rho(B)$.
- $\mathsf{c}(H) = \mathsf{c}(B)$, apart from some extremal cases.

Commutative Krull monoids

A commutative monoid H is a Krull monoid
if one of the following equivalent statements holds:

(a) H is completely integrally closed and v-noetherian.

(b) There is a divisor homomorphism $\varphi\colon H \to F = \mathcal{F}(P)$
(For all $a, b \in H$: $a \mid b$ in $H \iff \varphi(a) \mid \varphi(b)$ in F)

(c) There is a divisor theory $\varphi\colon H \to F = \mathcal{F}(P)$

Examples:

- A domain R is Krull iff $R \setminus \{0\}$ is a Krull monoid.
- A v-Marot ring is Krull iff its monoid of regular elements is Krull.
- Regular congruence submonoids of Krull domains are Krull.
- Frisch, Reinhart: Monadic submonoids of $\mathrm{Int}(R)$, R factorial
- Facchini, 2002: Let $\mathcal{C}$ be a class of modules and $\mathcal{V}(\mathcal{C})$ the semigroup of isomorphism classes. If all $\mathrm{End}_R(M)$ are semilocal, then $\mathcal{V}(\mathcal{C})$ is Krull.

Zero-Sum Sequences I

Let $G = (G, +)$ be an abelian group and $G_0 \subset G$ a subset.

- A sequence $S = (g_1, \ldots, g_\ell)$ over G_0: finite, unordered sequence of terms from G_0, repetition allowed.
- S has sum zero if $\sigma(S) = g_1 + \ldots + g_\ell = 0$.
- The set of (zero-sum) sequences forms a monoid with concatenation of sequences as the operation.

Formalization: Consider sequences as elements in $\mathcal{F}(G_0)$. Then

- $\mathcal{B}(G_0) = \{S \in \mathcal{F}(G_0) \mid \sigma(S) = 0\} \subset \mathcal{F}(G_0)$ is a submonoid.
- $\mathcal{B}(G_0) \hookrightarrow \mathcal{F}(G_0)$ is a Krull monoid, because

$$T \mid S \text{ in } \mathcal{B}(G_0) \quad \text{if and only if} \quad T \mid S \text{ in } \mathcal{F}(G_0)\,.$$

Zero-Sum Sequences II

Notation: $\mathcal{L}(G_0) := \mathcal{L}\big(\mathcal{B}(G_0)\big)$, $\Delta(G_0) := \Delta\big(\mathcal{B}(G_0)\big)$, $\rho(G_0) := \rho\big(\mathcal{B}(G_0)\big)$, and $\mathsf{c}(G_0) = \mathsf{c}\big(\mathcal{B}(G_0)\big)$.

The Krull monoid $\mathcal{B}(G_0)$ is studied with methods from Additive Combinatorics.

In particular, the Davenport constant

$$\mathsf{D}(G_0) := \sup\{|S| \colon S \text{ is a minimal zero-sum sequence over } G_0\}$$

is a well-studied invariant in Additive Combinatorics.

FACT 8. Let G be finite abelian.

- $2 + \max \Delta(G) \le \mathsf{c}(G) \le \mathsf{D}(G)$.
- $\rho(G) = \mathsf{D}(G)/2$.

The Davenport constant of finite abelian groups

Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ where $1 < n_1 \mid \ldots \mid n_r$. Then

$$\mathsf{D}^*(G) := 1 + \sum_{i=1}^{r} (n_i - 1) \le \mathsf{D}(G) \le |G| \,.$$

- Olson, Kruyswijk, 1960s: Equality (on the left) for p-groups and rank 2 groups.
- G. + Schneider, 1992: Inequality (on the left) can be strict for rank four groups on.
- Girard, 2018: For every $r \in \mathbb{N}$, $\lim_{n \to \infty} \frac{\mathsf{D}(C_n^r)}{rn} = 1$.
- Girard + Schmid, 2019: Progress on $\mathsf{D}(G)$ and on the Erdős-Ginzburg-Ziv constant $\mathsf{s}(G)$, mainly for rank three groups.
- Chao Liu, 2019: New lower bounds for $\mathsf{D}(G)$.

Transfer hom. from a commutative Krull monoid to $\mathcal{B}(G_0)$

Suppose the embedding $H \hookrightarrow \mathcal{F}(P)$ is a divisor theory.

$$\begin{array}{ccc} H & \longrightarrow & \mathcal{F}(P) \cong \mathcal{I}_v^*(H) \\ \boldsymbol{\beta} \Big\downarrow & & \Big\downarrow \widetilde{\boldsymbol{\beta}} \\ \mathcal{B}(G_0) & \longrightarrow & \mathcal{F}(G_0) \end{array}$$

Then $\widetilde{\boldsymbol{\beta}}$ and its restriction $\boldsymbol{\beta} = \widetilde{\boldsymbol{\beta}} \mid H$ are transfer homomorphisms mapping

$$a = p_1 \cdot \ldots \cdot p_l \in \mathcal{F}(P) \quad \text{to} \quad S = \boldsymbol{\beta}(a) = [p_1] \cdot \ldots \cdot [p_l] \in \mathcal{F}(G_0)$$

Transfer Krull monoids: Definition

A monoid H is said to be a transfer Krull monoid if one of the following equivalent statements holds:

(a) There is a commutative Krull monoid B and a transfer homomorphism $\theta\colon H \to B$.

(b) There is an abelian group G, a subset $G_0 \subset G$, and a transfer homomorphism $\theta\colon H \to \mathcal{B}(G_0)$.

H is said to be of of finite type if there is a finite G_0 such that ....

Note:

- (Easy) Commutative Krull monoids are transfer Krull.
- (Not easy) There are many others.

Transfer Krull but not necessarily Krull: Half-factorial monoids

If H is half-factorial, then the map

$$\theta\colon H \to \mathcal{B}(\{0\}), \qquad \theta(a) = \begin{cases} 1 & a \text{ is a unit} \\ 0 & a \text{ is an atom} \end{cases}$$

is a transfer homomorphism.

- All half-factorial monoids are transfer Krull.
- Roitman, 2016: Half-factorial domains need not be Mori.

Transfer Krull but not Krull: $(K+\mathfrak{m})$-Domains

A $(K+\mathfrak{m})$-domain is the sum of a field and a maximal ideal: e.g., $K+XL[X]$ or $K+XL[\![X]\!]$.

Proposition

If $R=K+\mathfrak{m}\subsetneq D=L+\mathfrak{m}$, *where* $\mathfrak{m}\in\max(D)$, *then*

$$D=L^{\times}R=D^{\times}R,\ D^{\times}\cap R=R^{\times},\quad \text{and}\quad (R:D)=\mathfrak{m}\,. \qquad (*)$$

Proposition

Let $R\subset D$ *be commutative domains with* $\mathsf{q}(R)=\mathsf{q}(D)$ *and* $(*)$.

1. *The embedding* $R^{\bullet}\hookrightarrow D^{\bullet}$ *is a transfer homomorphism.*
2. *If* D *is Krull, then* R *is transfer Krull.*

"Good" seminormal weakly Krull are transfer Krull

Theorem (G. + Kainrath + Reinhart, 2015)

Let H be seminormal v-noetherian weakly Krull with nontrivial conductor $\mathfrak{f}$, finite v-class group $\mathcal{C}_v(H)$, and suppose that every class contains $\mathfrak{p} \in \mathfrak{X}(H)$ with $\mathfrak{p} \not\supset \mathfrak{f}$.

Note: *Seminormal orders in number fields have all these properties.*

Suppose that

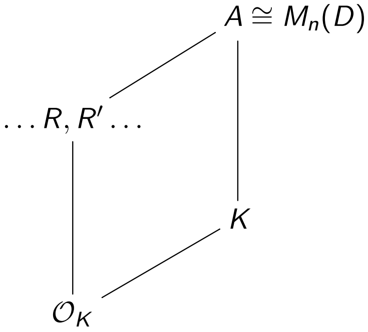
- *The natural map $\mathfrak{X}(\widehat{H}) \to \mathfrak{X}(H)$ is bijective*
- *$\overline{\vartheta}\colon \mathcal{C}_v(H) \to \mathcal{C}_v(\widehat{H})$ is an isomorphism*

Then there is a transfer homomorphism $\theta\colon H \to \mathcal{B}(\mathcal{C}_v(H))$.

In particular, H is a transfer Krull monoid.

Classical Maximal Orders: Smertnig, 2013

- A be a central simple K-algebra,
- $\mathcal{O}_K$ a holomorphy ring of K,
- and R a classical maximal $\mathcal{O}_K$-order in A (R subring of A, $Z(R) = \mathcal{O}_K$, f.g. as $\mathcal{O}_K$-module, maximal).
- e.g., $R = M_n(\mathcal{O}_K)$



Then

- R is transfer Krull $\iff$ if every stably free left R-ideal is free.
- If this holds, then there exists a transfer homomorphism

$$\theta \colon R^\bullet \to \mathcal{B}(\mathcal{C}_A(\mathcal{O}_K)),$$

with $\mathcal{C}_A(\mathcal{O}_K)$ a ray class group of $\mathcal{O}_K$.

Non-commutative rings

Theorem

- *Baeth + Smertnig, 2015: Bounded Dedekind prime rings, whose stably free left ideals are free, are transfer Krull and the transfer homomorphism respects catenary degrees.*
  Method: Theory of one-sided divisorial ideals
- *Smertnig, 2019: Bounded HNP rings, whose stably free left ideals are free, are transfer Krull.*
  Method: Theory of f.g. projective modules over HNP rings, as established by Levy and Robson

Not transfer Krull I

A commutative monoid H is strongly primary if $\mathfrak{m} = H \setminus H^\times \neq \emptyset$ and for every $a \in \mathfrak{m}$ there is $n \in \mathbb{N}$ such that $\mathfrak{m}^n \subset aH$.

- An additive submonoid $H \subset (\mathbb{N}_0^s, +)$ is primary iff its cone $\mathcal{C}(H) \setminus \{0\}$ is open.
- Every finitely generated monoid is Mori (i.e, v-noetherian).
- Every primary Mori monoid is strongly primary.
- Numerical monoids are strongly primary.
- Every one-dimensional local Mori domain is strongly primary.

Theorem (G. + Schmid + Zhong, 2016)

Strongly primary monoids, that are not half-factorial, are NOT transfer Krull.

Not transfer Krull II

- Frisch et al.: If R is Dedekind with infinitely many maximal ideals of finite index, then $\mathrm{Int}(R)$ is not transfer Krull.
- Facchini et al.: The monoid of polynomials with non-negative integer coeficients is not transfer Krull.
- Oh: The monoid $\mathcal{B}(G)$ of product-one sequences over a finite group G is transfer Krull if and only if G is abelian if and only if it is Krull.
- F. Gotti: Additive submonoids of $(\mathbb{Q}_{\geq 0}, +)$, that are not isomorphic to $(\mathbb{N}_0, +)$, are not transfer Krull.
- G. + Schwab: $B_n = \langle a, b \mid ba = b^n \rangle$ is not transfer Krull.
- Fan + Tringali: Power monoids are not transfer Krull (e.g., the monoid of finite subsets of $(\mathbb{N}_0, +)$ with set addition).

Outline

Sets of arithmetical invariants

Transfer Krull monoids

Main Results

Open Problems

Realization results for sets of lengths

The following monoids and domains contain every finite subset $L \subset \mathbb{N}_{\geq 2}$ as a set of lengths:

- Kainrath, 1999: H is a Krull monoid with infinite class group having prime divsors in all classes.
- Frisch et al., 2019: $\text{Int}(R)$, where R is a Dedekind domain having infinitely many maximal ideals of finite index.
- F. Gotti, 2019: Some primary submonoids of $(\mathbb{Q}_{\geq 0}, +)$.

Transfer Krull monoids are fully elastic

A monoid H is fully elastic if for every rational number q with $1 \le q < \rho(H)$ there is an $L \in \mathcal{L}(H)$ with $\rho(L) = q$.

Theorem (G. + Zhong, 2019)

Every transfer Krull monoid is fully elastic.

FACT 9. If H is transfer Krull of finite type, then $\rho(H) < \infty$.

Corollary

Let H be a strongly primary monoid that is not half-factorial. There is a $\beta \in \mathbb{Q}_{\ge 1}$ such that $\rho(L) \ge \beta$ for all $L \in \mathcal{L}(H)$ with $\rho(L) > 1$. In particular, H is not transfer Krull.

Sets of Distances and Sets of Catenary Degrees

Theorem (G. + Zhong, 2019)

Let H be transfer Krull with transfer homomorphism $\theta\colon H \to \mathcal{B}(G)$ to an abelian group G and a distance d with $\mathsf{c}_{\mathsf{d}}(H, \theta) \le 2$.

1. *If $\mathsf{D}(G) \le 2$, then $\Delta(H) = \emptyset$.*
2. *If $\mathsf{D}(G) = 3$, then $\Delta(H) = \{1\}$.*
3. *Suppose that G is finite with $\mathsf{D}(G) \ge 4$. Then $\Delta(H)$ and $\mathsf{Ca}_{\mathsf{d}}(H)$ are intervals. If $\mathsf{D}(G) = \mathsf{D}^*(G)$, then*

$$\big(2 + \Delta(H)\big) \cup \{2\} = \mathsf{Ca}_{\mathsf{d}}(H) = [2, \mathsf{c}_{\mathsf{d}}(H)]\,.$$

4. *If G is infinite, then $\Delta(H) = \mathbb{N}$ and $\mathsf{Ca}_{\mathsf{d}}(H) = \mathbb{N}_{\ge 2}$.*

On the catenary degree

Let H be a Krull monoid with class group G having prime divisors in all classes, say

$$|G| \ge 3 \quad \text{and} \quad G \cong C_{n_1} \oplus \ldots \oplus C_{n_r} \quad \text{with} \quad 1 < n_1 \mid \ldots \mid n_r \,.$$

Then $\mathsf{c}(H) = \mathsf{c}(G)$ and we have

- $$\max\left\{ n_r, 1 + \sum_{i=1}^{r} \left\lfloor \frac{n_i}{2} \right\rfloor \right\} \le \mathsf{c}(G) \le \mathsf{D}(G)\,.$$
- $\mathsf{c}(G) = \mathsf{D}(G)$ iff G is cyclic or an elementary 2-group.
- $\mathsf{c}(G) = \mathsf{D}(G) - 1$ iff $G \cong C_2 \oplus C_{2n}$ or $G \cong C_2^{r-1} \oplus C_4$.

Without assumption on the distribution on prime divisors

Theorem

- *Fan + G., 2019: For every finite nonempty subset $C \subset \mathbb{N}_{\geq 2}$ there is a finitely generated Krull monoid with finite class group such that $\mathsf{Ca}(H) = C$.*
- *G. + Schmid, 2017: For every finite nonempty subset $\Delta \subset \mathbb{N}$ with $\min \Delta = \gcd \Delta$ there is a finitely generated Krull monoid H such that $\Delta(H) = \Delta$.*

Fact 10. By Claborn's Realization Theorem, H can be chosen to be the multiplicative monoid of a Dedekind domain.

Tame degrees I

The tame degree $\mathsf{t}(a, u)$ of an element $a \in H$ and an atom $u \in H$ is the smallest integer N with the following property:

If $a \in uH$, then for any factorization $a = v_1 \cdot \ldots \cdot v_n$,
there is a subproduct which is a multiple of u,
and a refactorization of this subproduct which contains u, say

$$v_1 \cdot \ldots \cdot v_m = u u_2 \cdot \ldots \cdot u_\ell \,,$$

such that

$$\max\{\ell, m\} \le N \,.$$

This means that we need to exchange at most N old atoms v_i
by at most N new atoms u_j to get a factorization
where the given u pops up.

Tame degrees II

- $\mathsf{t}(a, u) \le \max \mathsf{L}(a)$.
- $\mathsf{Ta}(H) = \{\mathsf{t}(a, u) \mid a \in H, u \in \mathcal{A}(H_{\mathrm{red}}), \mathsf{t}(a, u) > 0\} \subset \mathbb{N}_0$ the set of tame degrees of H.
- H is locally tame if

$$\mathsf{t}(H, u) = \sup\{\mathsf{t}(a, u) \mid a \in H\} < \infty\,.$$

- $\mathsf{t}(H) = \sup \mathsf{Ta}(H) = \sup\{\mathsf{t}(H, u) \mid u \in \mathcal{A}(H_{\mathrm{red}})\} \in \mathbb{N}_0 \cup \{\infty\}$ is the tame degree of H.
- The following are locally tame:
  - Krull monoids with finite class group
  - One-dimensional local Mori domains

Tame degrees III

Theorem (G. + Zhong, 2019)

Let H be a commutative Krull monoid with class group G such that $\mathsf{D}(G) \ge 3$ and suppose that every class contains a prime divisor.

- $\mathsf{Ta}(C_3) = \mathsf{Ta}(C_2 \oplus C_2) = \{3\}$. *If G is finite and either $\mathsf{D}(G) \ge 4$ or there is a nonzero class containing at least two distinct prime divisors, then $[2, \mathsf{D}(G)] \subset \mathsf{Ta}(H)$.*
- *If every class contains at least $\mathsf{D}(G) + 1$ prime divisors, then $\mathsf{Ta}(H) = [2, \mathsf{t}(H)]$.*
- *If G is infinite, then $\mathsf{Ta}(H) = \mathbb{N}_{\ge 2}$.*

Tame degrees IV

Theorem

Let H be a commutative Krull monoid whose class group $G = C_2^r$ with $r \in \mathbb{N}_0$, and suppose that every class contains a prime divisor.

1. *If $r = 0$ then $\mathsf{Ta}(H) = \mathsf{Ta}(G) = \emptyset$, and if $r = 1$ then $\mathsf{Ta}(H) = \{2\}$ and $\mathsf{Ta}(G) = \emptyset$.*
2. *If $r = 2$ then $\mathsf{Ta}(G) = \{3\}$, and if one nonzero class contains at least two distinct prime divisors, then $\mathsf{Ta}(H) = [2, 3]$.*
3. *If $r = 3$ then $\mathsf{Ta}(G) = [2, 4]$, and if one nonzero class contains at least two distinct prime divisors, then $\mathsf{Ta}(H) = [2, 5]$.*
4. $$\mathsf{Ta}(H) \begin{cases} = \mathsf{Ta}(G) = [2, 1 + \frac{r^2}{2}] & \text{if } r \geq 4 \text{ is even,} \\ \supset \mathsf{Ta}(G) \supset [2, 2 + \frac{r(r-1)}{2}] & \text{if } r \geq 5 \text{ is odd.} \end{cases}$$

Tame degrees V

Lemma

For every finite nonempty subset $C \subset \mathbb{N}_{\ge 2}$ *there is a finitely generated commutative Krull monoid* H *with finite class group such that* $\mathsf{Ta}(H) = C$.

Outline

Which orders are transfer Krull?

Let R be an order in an algebraic number field.

Problem 1. Characterize when R

(a) is transfer Krull.

(b) is transfer Krull of finite type: this means there is a group G, a finite $G_0 \subset G$, and a transfer hom. $\theta \colon R^\bullet \to \mathcal{B}(G_0)$.

(c) admits a transfer homomorphism $\theta \colon R^\bullet \to \mathcal{B}(G)$, where G is a finite abelian group.

Note

- There are half-factorial orders in quadratic number fields that are not seminormal.
- If (b) holds, then $\mathfrak{X}(\overline{R}) \to \mathfrak{X}(R)$, $\mathfrak{p} \mapsto \mathfrak{p} \cap R$, is bijective.
- If (c) holds, then $\min \Delta(R) = 1$, but (G. + Reinhart 2019)

$$\min \Delta\big(\mathbb{Z} \oplus 5\sqrt{15}\mathbb{Z}\big) = 2\,.$$

On the maximum of the set of distances

Let H be a Krull monoid with finite class group G and prime divisors in all classes and suppose that

$$\mathsf{D}(G) = \mathsf{D}^*(G) \geq 3\,.$$

Then

$$\big(2 + \Delta(G)\big) \cup \{2\} = \mathsf{Ca}(G) = [2, \mathsf{c}(G)]\,.$$

Problem 2. Prove that

(a) $\mathsf{c}(C_p \oplus C_p) = p$ for all odd primes p.

(b) $\mathsf{c}(C_3^r) = r + 1$.

On tame degrees

Let H be a Krull monoid with finite class group G, $\mathsf{D}(G) \geq 3$, and suppose that each class contains at least $\mathsf{D}(G)+1$ prime divisors. Then

- $\mathsf{Ta}(H) = [2, \mathsf{t}(H)]$.
- $\mathsf{D}(G) \leq \mathsf{t}(G) \leq \mathsf{t}(H) \leq \frac{1+\mathsf{D}(G)(\mathsf{D}(G)-1)}{2}$.

Problem 3.

(a) Is $\mathsf{t}(G) = \mathsf{t}(H)$ for almost all groups?

(b) The only groups for which $\mathsf{t}(G)$ is known are elementary 2-groups of even rank.
Determine $\mathsf{t}(C_2^r)$ for odd $r \geq 5$.

Invariants of numerical monoids

We know that

- G. + Schmid, 2018: For every finite $L \subset \mathbb{N}_{\geq 2}$ there is a numerical monoid H with $L \in \mathcal{L}(H)$.
- O'Neill + Pelayo, 2018: For every finite $C \subset \mathbb{N}_{\geq 2}$ there is a numerical monoid with $\mathsf{Ca}(H) = C$.
- Colton + Kaplan, 2017: For every two-element set Δ there is a numerical monoid with $\Delta(H) = \Delta$.

Problem 4.

- Study the set of tame degrees $\mathsf{Ta}(H)$ for numerical monoids and prove a realization theorem.
- Is every finite set Δ with $\min \Delta = \gcd \Delta$ equal to the set of distances $\Delta(H)$ for some numerical monoid H?

Conference Announcement

Algebra Conference: Rings and ... 2020

organized by

Sophie Frisch and her team

July 20 – 24, 2020

Graz University of Technology, Austria