On the Atomicity of Monoid Algebras of Finite Characteristic

(joint work with Jim Coykendall)

Felix Gotti

UC Berkeley

AMS Special Session:
Factorization and Arithmetic Properties
of Integral Domains and Monoids

Honolulu HI

March 24

Outline

Outline

Outline

Outline

Monoids

Definition (monoid)

Just for today, a semigroup $(M, *)$ with identity e is called a *monoid* provided that it is

1. commutative;
2. cancellative;
3. torsion-free (i.e., $x^n = y^n$ implies $x = y$ for all $n \in \mathbb{N}$, $x, y \in M$.)

For a monoid M, we let $M^\times$ denote the set of invertible elements (or units) of M.

Notation: From now on, monoids here will be additively written unless otherwise specified.

Monoids

Definition (monoid)

Just for today, a semigroup $(M, *)$ with identity e is called a *monoid* provided that it is

1. commutative;
2. cancellative;
3. torsion-free (i.e., $x^n = y^n$ implies $x = y$ for all $n \in \mathbb{N}$, $x, y \in M$.)

For a monoid M, we let $M^\times$ denote the set of invertible elements (or units) of M.

Notation: From now on, monoids here will be additively written unless otherwise specified.

Monoids

Definition (monoid)

Just for today, a semigroup $(M, *)$ with identity e is called a *monoid* provided that it is

1. commutative;
2. cancellative;
3. torsion-free (i.e., $x^n = y^n$ implies $x = y$ for all $n \in \mathbb{N}$, $x, y \in M$.)

For a monoid M, we let $M^\times$ denote the set of invertible elements (or units) of M.

Notation: From now on, monoids here will be additively written unless otherwise specified.

Atomic Monoids

Let M be a monoid.

- An element $a \in M \setminus M^\times$ is an *atom* if $x + y = a$ implies that either $x \in M^\times$ or $y \in M^\times$.
- We let $\mathcal{A}(M)$ denote the set of atoms of M.
- The monoid M is called *atomic* if every element in $M \setminus M^\times$ can be expressed as a sum of atoms.

Proposition (Easy to verify)

Let M and N be monoids. If M and N are atomic, then $M \times N$ is atomic.

Atomic Monoids

Let M be a monoid.

- An element $a \in M \setminus M^\times$ is an *atom* if $x + y = a$ implies that either $x \in M^\times$ or $y \in M^\times$.
- We let $\mathcal{A}(M)$ denote the set of atoms of M.
- The monoid M is called *atomic* if every element in $M \setminus M^\times$ can be expressed as a sum of atoms.

Proposition (Easy to verify)

Let M and N be monoids. If M and N are atomic, then $M \times N$ is atomic.

Atomic Monoids

Let M be a monoid.

- An element $a \in M \setminus M^\times$ is an *atom* if $x + y = a$ implies that either $x \in M^\times$ or $y \in M^\times$.
- We let $\mathcal{A}(M)$ denote the set of atoms of M.
- The monoid M is called *atomic* if every element in $M \setminus M^\times$ can be expressed as a sum of atoms.

Proposition (Easy to verify)

Let M and N be monoids. If M and N are atomic, then $M \times N$ is atomic.

Atomic Monoids

Let M be a monoid.

- An element $a \in M \setminus M^\times$ is an *atom* if $x + y = a$ implies that either $x \in M^\times$ or $y \in M^\times$.
- We let $\mathcal{A}(M)$ denote the set of atoms of M.
- The monoid M is called *atomic* if every element in $M \setminus M^\times$ can be expressed as a sum of atoms.

Proposition (Easy to verify)

Let M and N be monoids. If M and N are atomic, then $M \times N$ is atomic.

ACCP Monoids

Let M be a monoid.

- A subset I of M is an *ideal* of M if $I + M \subseteq I$.
- The ideal I is *principal* if $I = x + M$ for some $x \in M$.
- The monoid M is an *ACCP monoid* if every ascending chain of principal ideals of M eventually stabilizes.

Proposition (Easy to prove)

Let M and N be monoids.

- *If M and N satisfy the ACCP, then $M \times N$ satisfies the ACCP.*
- *If M is an ACCP monoid, then M is atomic.*

Example. Let p_n denote the n^{th} odd prime. The Gram's monoid,

$$G = \left\langle \frac{1}{2^n \cdot p_n} \;\middle|\; n \in \mathbb{N} \right\rangle$$

is atomic but does not satisfy the ACCP as the ascending chain of principal ideals $\{1/2^n + G\}$ does not stabilize.

ACCP Monoids

Let M be a monoid.

- A subset I of M is an *ideal* of M if $I + M \subseteq I$.
- The ideal I is *principal* if $I = x + M$ for some $x \in M$.
- The monoid M is an *ACCP monoid* if every ascending chain of principal ideals of M eventually stabilizes.

Proposition (Easy to prove)

Let M and N be monoids.

- *If M and N satisfy the ACCP, then $M \times N$ satisfies the ACCP.*
- *If M is an ACCP monoid, then M is atomic.*

Example. Let p_n denote the n^{th} odd prime. The Gram's monoid,

$$G = \left\langle \frac{1}{2^n \cdot p_n} \;\middle|\; n \in \mathbb{N} \right\rangle$$

is atomic but does not satisfy the ACCP as the ascending chain of principal ideals $\{1/2^n + G\}$ does not stabilize.

ACCP Monoids

Let M be a monoid.

- A subset I of M is an *ideal* of M if $I + M \subseteq I$.
- The ideal I is *principal* if $I = x + M$ for some $x \in M$.
- The monoid M is an *ACCP monoid* if every ascending chain of principal ideals of M eventually stabilizes.

Proposition (Easy to prove)

Let M and N be monoids.

- *If M and N satisfy the ACCP, then $M \times N$ satisfies the ACCP.*
- *If M is an ACCP monoid, then M is atomic.*

Example. Let p_n denote the n^{th} odd prime. The Gram's monoid,

$$G = \left\langle \frac{1}{2^n \cdot p_n} \;\middle|\; n \in \mathbb{N} \right\rangle$$

is atomic but does not satisfy the ACCP as the ascending chain of principal ideals $\{1/2^n + G\}$ does not stabilize.

ACCP Monoids

Let M be a monoid.

- A subset I of M is an *ideal* of M if $I + M \subseteq I$.
- The ideal I is *principal* if $I = x + M$ for some $x \in M$.
- The monoid M is an *ACCP monoid* if every ascending chain of principal ideals of M eventually stabilizes.

Proposition (Easy to prove)

Let M and N be monoids.

- *If M and N satisfy the ACCP, then $M \times N$ satisfies the ACCP.*
- *If M is an ACCP monoid, then M is atomic.*

Example. Let p_n denote the n^{th} odd prime. The Gram's monoid,

$$G = \left\langle \frac{1}{2^n \cdot p_n} \;\middle|\; n \in \mathbb{N} \right\rangle$$

is atomic but does not satisfy the ACCP as the ascending chain of principal ideals $\{1/2^n + G\}$ does not stabilize.

ACCP Monoids

Let M be a monoid.

- A subset I of M is an *ideal* of M if $I + M \subseteq I$.
- The ideal I is *principal* if $I = x + M$ for some $x \in M$.
- The monoid M is an *ACCP monoid* if every ascending chain of principal ideals of M eventually stabilizes.

Proposition (Easy to prove)

Let M and N be monoids.

- *If M and N satisfy the ACCP, then $M \times N$ satisfies the ACCP.*
- *If M is an ACCP monoid, then M is atomic.*

Example. Let p_n denote the n^{th} odd prime. The Gram's monoid,

$$G = \left\langle \frac{1}{2^n \cdot p_n} \;\middle|\; n \in \mathbb{N} \right\rangle$$

is atomic but does not satisfy the ACCP as the ascending chain of principal ideals $\{1/2^n + G\}$ does not stabilize.

ACCP Monoids

Let M be a monoid.

- A subset I of M is an *ideal* of M if $I + M \subseteq I$.
- The ideal I is *principal* if $I = x + M$ for some $x \in M$.
- The monoid M is an *ACCP monoid* if every ascending chain of principal ideals of M eventually stabilizes.

Proposition (Easy to prove)

Let M and N be monoids.

- *If M and N satisfy the ACCP, then $M \times N$ satisfies the ACCP.*
- *If M is an ACCP monoid, then M is atomic.*

Example. Let p_n denote the n^{th} odd prime. The Gram's monoid,

$$G = \left\langle \frac{1}{2^n \cdot p_n} \;\middle|\; n \in \mathbb{N} \right\rangle$$

is atomic but does not satisfy the ACCP as the ascending chain of principal ideals $\{1/2^n + G\}$ does not stabilize.

Rank of a Monoid

Definition (Grothendieck group and rank)

Let M be a monoid.

- The *Grothendieck group* $\mathrm{gp}(M)$ of M is the abelian group satisfying that any abelian group containing a homomorphic image of M will also contain a homomorphic image of $\mathrm{gp}(M)$.
- The *rank* of M is the rank of the group $\mathrm{gp}(M)$, that is, the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{gp}(M)$.

Example. For a submonoid M of $(\mathbb{Q}_{\geq 0}, +)$ we have that $\mathrm{gp}(M) \cong \{r - s \mid r, s \in M\}$ and so M has rank 1.

Example. If α and $\beta \in \mathbb{R}_{>0} \setminus \mathbb{Q}$ are linearly independent over $\mathbb{Q}$ and M_1 and M_2 are submonoids of $(\mathbb{Q}_{\geq 0}, +)$, then

$$\mathrm{gp}(\alpha M_1 + \beta M_2) \cong \alpha \mathrm{gp}(M_1) \oplus \beta \mathrm{gp}(M_2),$$

and so $\alpha M_1 \oplus \beta M_2$ has rank 2.

Rank of a Monoid

Definition (Grothendieck group and rank)

Let M be a monoid.

- The *Grothendieck group* $\mathrm{gp}(M)$ of M is the abelian group satisfying that any abelian group containing a homomorphic image of M will also contain a homomorphic image of $\mathrm{gp}(M)$.
- The *rank* of M is the rank of the group $\mathrm{gp}(M)$, that is, the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{gp}(M)$.

Example. For a submonoid M of $(\mathbb{Q}_{\geq 0}, +)$ we have that $\mathrm{gp}(M) \cong \{r - s \mid r, s \in M\}$ and so M has rank 1.

Example. If α and $\beta \in \mathbb{R}_{>0} \setminus \mathbb{Q}$ are linearly independent over $\mathbb{Q}$ and M_1 and M_2 are submonoids of $(\mathbb{Q}_{\geq 0}, +)$, then

$$\mathrm{gp}(\alpha M_1 + \beta M_2) \cong \alpha \mathrm{gp}(M_1) \oplus \beta \mathrm{gp}(M_2),$$

and so $\alpha M_1 \oplus \beta M_2$ has rank 2.

Rank of a Monoid

Definition (Grothendieck group and rank)

Let M be a monoid.

- The *Grothendieck group* $\mathrm{gp}(M)$ of M is the abelian group satisfying that any abelian group containing a homomorphic image of M will also contain a homomorphic image of $\mathrm{gp}(M)$.
- The *rank* of M is the rank of the group $\mathrm{gp}(M)$, that is, the dimension of the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{gp}(M)$.

Example. For a submonoid M of $(\mathbb{Q}_{\ge 0}, +)$ we have that $\mathrm{gp}(M) \cong \{r - s \mid r, s \in M\}$ and so M has rank 1.

Example. If α and $\beta \in \mathbb{R}_{>0} \setminus \mathbb{Q}$ are linearly independent over $\mathbb{Q}$ and M_1 and M_2 are submonoids of $(\mathbb{Q}_{\ge 0}, +)$, then

$$\mathrm{gp}(\alpha M_1 + \beta M_2) \cong \alpha \mathrm{gp}(M_1) \oplus \beta \mathrm{gp}(M_2),$$

and so $\alpha M_1 \oplus \beta M_2$ has rank 2.

Puiseux Monoids

Definition (Puiseux monoid)

A *Puiseux monoid* is an additive submonoid of $(\mathbb{Q}_{\geq 0}, +)$.

Elementary Facts:

- A Puiseux monoid M is isomorphic to a submonoid of $(\mathbb{N}_0, +)$ iff M is finitely generated.
- If a submonoid of $(\mathbb{Q}, +)$ is not a group, then it is isomorphic to a Puiseux monoid.
- The only homomorphisms of Puiseux monoids are given by rational multiplication.
- A monoid has rank 1 if and only if it is isomorphic to a Puiseux monoid.
- If 0 is not a limit point of a Puiseux monoid, then the monoid is atomic.
- Not every Puiseux monoid is atomic: $\langle 1/2^n \mid n \in \mathbb{N} \rangle$ has no atoms.

Puiseux Monoids

Definition (Puiseux monoid)

A *Puiseux monoid* is an additive submonoid of $(\mathbb{Q}_{\geq 0}, +)$.

Elementary Facts:

- A Puiseux monoid M is isomorphic to a submonoid of $(\mathbb{N}_0, +)$ iff M is finitely generated.
- If a submonoid of $(\mathbb{Q}, +)$ is not a group, then it is isomorphic to a Puiseux monoid.
- The only homomorphisms of Puiseux monoids are given by rational multiplication.
- A monoid has rank 1 if and only if it is isomorphic to a Puiseux monoid.
- If 0 is not a limit point of a Puiseux monoid, then the monoid is atomic.
- Not every Puiseux monoid is atomic: $\langle 1/2^n \mid n \in \mathbb{N} \rangle$ has no atoms.

Puiseux Monoids

Definition (Puiseux monoid)

A *Puiseux monoid* is an additive submonoid of $(\mathbb{Q}_{\geq 0}, +)$.

Elementary Facts:

- A Puiseux monoid M is isomorphic to a submonoid of $(\mathbb{N}_0, +)$ iff M is finitely generated.
- If a submonoid of $(\mathbb{Q}, +)$ is not a group, then it is isomorphic to a Puiseux monoid.
- The only homomorphisms of Puiseux monoids are given by rational multiplication.
- A monoid has rank 1 if and only if it is isomorphic to a Puiseux monoid.
- If 0 is not a limit point of a Puiseux monoid, then the monoid is atomic.
- Not every Puiseux monoid is atomic: $\langle 1/2^n \mid n \in \mathbb{N} \rangle$ has no atoms.

Puiseux Monoids

Definition (Puiseux monoid)

A *Puiseux monoid* is an additive submonoid of $(\mathbb{Q}_{\geq 0}, +)$.

Elementary Facts:

- A Puiseux monoid M is isomorphic to a submonoid of $(\mathbb{N}_0, +)$ iff M is finitely generated.
- If a submonoid of $(\mathbb{Q}, +)$ is not a group, then it is isomorphic to a Puiseux monoid.
- The only homomorphisms of Puiseux monoids are given by rational multiplication.
- A monoid has rank 1 if and only if it is isomorphic to a Puiseux monoid.
- If 0 is not a limit point of a Puiseux monoid, then the monoid is atomic.
- Not every Puiseux monoid is atomic: $\langle 1/2^n \mid n \in \mathbb{N} \rangle$ has no atoms.

Puiseux Monoids

Definition (Puiseux monoid)

A *Puiseux monoid* is an additive submonoid of $(\mathbb{Q}_{\geq 0}, +)$.

Elementary Facts:

- A Puiseux monoid M is isomorphic to a submonoid of $(\mathbb{N}_0, +)$ iff M is finitely generated.
- If a submonoid of $(\mathbb{Q}, +)$ is not a group, then it is isomorphic to a Puiseux monoid.
- The only homomorphisms of Puiseux monoids are given by rational multiplication.
- A monoid has rank 1 if and only if it is isomorphic to a Puiseux monoid.
- If 0 is not a limit point of a Puiseux monoid, then the monoid is atomic.
- Not every Puiseux monoid is atomic: $\langle 1/2^n \mid n \in \mathbb{N} \rangle$ has no atoms.

Puiseux Monoids

Definition (Puiseux monoid)

A *Puiseux monoid* is an additive submonoid of $(\mathbb{Q}_{\geq 0}, +)$.

Elementary Facts:

- A Puiseux monoid M is isomorphic to a submonoid of $(\mathbb{N}_0, +)$ iff M is finitely generated.
- If a submonoid of $(\mathbb{Q}, +)$ is not a group, then it is isomorphic to a Puiseux monoid.
- The only homomorphisms of Puiseux monoids are given by rational multiplication.
- A monoid has rank 1 if and only if it is isomorphic to a Puiseux monoid.
- If 0 is not a limit point of a Puiseux monoid, then the monoid is atomic.
- Not every Puiseux monoid is atomic: $\langle 1/2^n \mid n \in \mathbb{N} \rangle$ has no atoms.

Puiseux Monoids

Definition (Puiseux monoid)

A *Puiseux monoid* is an additive submonoid of $(\mathbb{Q}_{\geq 0}, +)$.

Elementary Facts:

- A Puiseux monoid M is isomorphic to a submonoid of $(\mathbb{N}_0, +)$ iff M is finitely generated.
- If a submonoid of $(\mathbb{Q}, +)$ is not a group, then it is isomorphic to a Puiseux monoid.
- The only homomorphisms of Puiseux monoids are given by rational multiplication.
- A monoid has rank 1 if and only if it is isomorphic to a Puiseux monoid.
- If 0 is not a limit point of a Puiseux monoid, then the monoid is atomic.
- Not every Puiseux monoid is atomic: $\langle 1/2^n \mid n \in \mathbb{N} \rangle$ has no atoms.

Why Should We Care About Puiseux Monoids?

Remark. Puiseux monoids allow us to construct useful examples of monoid algebras. Such algebras have been used to:

1. disprove Cohn's conjecture that any atomic domain must satisfy the ACCP (A. Grams);
2. construct the first example of two-dimensional non-Noetherian UFD (R. Gilmer);
3. find an ACCP domain with a localization which is not an ACCP domain (D. Anderson, D. Anderson, and M. Zafrullah);
4. construct non-atomic monoid algebras with atomic exponent monoids (J. Coykendall and G.).

Why Should We Care About Puiseux Monoids?

Remark. Puiseux monoids allow us to construct useful examples of monoid algebras. Such algebras have been used to:

1. disprove Cohn's conjecture that any atomic domain must satisfy the ACCP (A. Grams);
2. construct the first example of two-dimensional non-Noetherian UFD (R. Gilmer);
3. find an ACCP domain with a localization which is not an ACCP domain (D. Anderson, D. Anderson, and M. Zafrullah);
4. construct non-atomic monoid algebras with atomic exponent monoids (J. Coykendall and G.).

Why Should We Care About Puiseux Monoids?

Remark. Puiseux monoids allow us to construct useful examples of monoid algebras. Such algebras have been used to:

1. disprove Cohn's conjecture that any atomic domain must satisfy the ACCP (A. Grams);
2. construct the first example of two-dimensional non-Noetherian UFD (R. Gilmer);
3. find an ACCP domain with a localization which is not an ACCP domain (D. Anderson, D. Anderson, and M. Zafrullah);
4. construct non-atomic monoid algebras with atomic exponent monoids (J. Coykendall and G.).

Why Should We Care About Puiseux Monoids?

Remark. Puiseux monoids allow us to construct useful examples of monoid algebras. Such algebras have been used to:

1. disprove Cohn's conjecture that any atomic domain must satisfy the ACCP (A. Grams);
2. construct the first example of two-dimensional non-Noetherian UFD (R. Gilmer);
3. find an ACCP domain with a localization which is not an ACCP domain (D. Anderson, D. Anderson, and M. Zafrullah);
4. construct non-atomic monoid algebras with atomic exponent monoids (J. Coykendall and G.).

Why Should We Care About Puiseux Monoids?

Remark. Puiseux monoids allow us to construct useful examples of monoid algebras. Such algebras have been used to:

1. disprove Cohn's conjecture that any atomic domain must satisfy the ACCP (A. Grams);
2. construct the first example of two-dimensional non-Noetherian UFD (R. Gilmer);
3. find an ACCP domain with a localization which is not an ACCP domain (D. Anderson, D. Anderson, and M. Zafrullah);
4. construct non-atomic monoid algebras with atomic exponent monoids (J. Coykendall and G.).

Monoid Domains/Algebras

Definition (monoid algebra/ring)

Let M be a monoid and let R be an integral domain. The ring $R[M]$ consisting of all the polynomial expressions on X with exponents in M and coefficients in R is called the *monoid domain* of M over R. When R is a field, $R[M]$ is called a *monoid algebra*.

Remark. Note that $R[\mathbb{N}_0]$ is the standard ring of polynomials over R, i.e., $R[\mathbb{N}_0] = R[X]$.

Observations. For an integral domain R and a monoid M, the following statements are easy to verify:

- $R[M]$ is an integral domain;
- The set of units of $R[M]$ is $R^\times$;
- If M is totally ordered, then $\deg(fg) = \deg f + \deg g$ for any $f, g \in R[M] \setminus \{0\}$.

Monoid Domains/Algebras

Definition (monoid algebra/ring)

Let M be a monoid and let R be an integral domain. The ring $R[M]$ consisting of all the polynomial expressions on X with exponents in M and coefficients in R is called the *monoid domain* of M over R. When R is a field, $R[M]$ is called a *monoid algebra*.

Remark. Note that $R[\mathbb{N}_0]$ is the standard ring of polynomials over R, i.e., $R[\mathbb{N}_0] = R[X]$.

Observations. For an integral domain R and a monoid M, the following statements are easy to verify:

- $R[M]$ is an integral domain;
- The set of units of $R[M]$ is $R^\times$;
- If M is totally ordered, then $\deg(fg) = \deg f + \deg g$ for any $f, g \in R[M] \setminus \{0\}$.

Monoid Domains/Algebras

Definition (monoid algebra/ring)

Let M be a monoid and let R be an integral domain. The ring $R[M]$ consisting of all the polynomial expressions on X with exponents in M and coefficients in R is called the *monoid domain* of M over R. When R is a field, $R[M]$ is called a *monoid algebra*.

Remark. Note that $R[\mathbb{N}_0]$ is the standard ring of polynomials over R, i.e., $R[\mathbb{N}_0] = R[X]$.

Observations. For an integral domain R and a monoid M, the following statements are easy to verify:

- $R[M]$ is an integral domain;
- The set of units of $R[M]$ is $R^\times$;
- If M is totally ordered, then $\deg(fg) = \deg f + \deg g$ for any $f, g \in R[M] \setminus \{0\}$.

Atomicity: A Question by Gilmer

Question (Gilmer, 1984)

For any pair (M, R) consisting of an atomic monoid M and an atomic integral domain R, is the monoid domain $R[M]$ atomic?

Theorem (Roitman)

There exists an atomic domain R such that $R[X]$ is not atomic.

Observations:

- The monoid-domain pairs found by Roitman are of the form $(\mathbb{N}_0, R)$. Observe that $\mathbb{N}_0$ is the "nicest" example of nontrivial atomic monoid.
- The "nicest" examples of atomic integral domains are fields.

Question (G'). Can we find a pair (M, F), where M is an atomic monoid and F is a field such that $F[M]$ is not atomic?

Atomicity: A Question by Gilmer

Question (Gilmer, 1984)

For any pair (M, R) consisting of an atomic monoid M and an atomic integral domain R, is the monoid domain $R[M]$ atomic?

Theorem (Roitman)

There exists an atomic domain R such that $R[X]$ is not atomic.

Observations:

- The monoid-domain pairs found by Roitman are of the form $(\mathbb{N}_0, R)$. Observe that $\mathbb{N}_0$ is the "nicest" example of nontrivial atomic monoid.
- The "nicest" examples of atomic integral domains are fields.

Question (G'). Can we find a pair (M, F), where M is an atomic monoid and F is a field such that $F[M]$ is not atomic?

Atomicity: A Question by Gilmer

Question (Gilmer, 1984)

For any pair (M, R) consisting of an atomic monoid M and an atomic integral domain R, is the monoid domain $R[M]$ atomic?

Theorem (Roitman)

There exists an atomic domain R such that $R[X]$ is not atomic.

Observations:

- The monoid-domain pairs found by Roitman are of the form $(\mathbb{N}_0, R)$. Observe that $\mathbb{N}_0$ is the "nicest" example of nontrivial atomic monoid.
- The "nicest" examples of atomic integral domains are fields.

Question (G'). Can we find a pair (M, F), where M is an atomic monoid and F is a field such that $F[M]$ is not atomic?

Atomicity: A Question by Gilmer

Question (Gilmer, 1984)

For any pair (M, R) consisting of an atomic monoid M and an atomic integral domain R, is the monoid domain $R[M]$ atomic?

Theorem (Roitman)

There exists an atomic domain R such that $R[X]$ is not atomic.

Observations:

- The monoid-domain pairs found by Roitman are of the form $(\mathbb{N}_0, R)$. Observe that $\mathbb{N}_0$ is the "nicest" example of nontrivial atomic monoid.
- The "nicest" examples of atomic integral domains are fields.

Question (G'). Can we find a pair (M, F), where M is an atomic monoid and F is a field such that $F[M]$ is not atomic?

Atomicity: A Question by Gilmer

Question (Gilmer, 1984)

For any pair (M, R) consisting of an atomic monoid M and an atomic integral domain R, is the monoid domain $R[M]$ atomic?

Theorem (Roitman)

There exists an atomic domain R such that $R[X]$ is not atomic.

Observations:

- The monoid-domain pairs found by Roitman are of the form $(\mathbb{N}_0, R)$. Observe that $\mathbb{N}_0$ is the "nicest" example of nontrivial atomic monoid.
- The "nicest" examples of atomic integral domains are fields.

Question (G'). Can we find a pair (M, F), where M is an atomic monoid and F is a field such that $F[M]$ is not atomic?

An Answer to G'

Theorem (Coykendall-G: 2018)

For each field F of finite characteristic, there exists an atomic monoid M such that $F[M]$ is not atomic.

Sketch of Proof:

1. Let $\{p_n\}$ be the strictly increasing sequence of primes. Set $p := \text{char}(F)$, and consider the Puiseux monoid
$$M_p := \langle \frac{1}{p^n p_n} \mid p_n \neq p \rangle.$$
Note that M_2 is the Gram's monoid.
2. The monoid M_p is atomic, but $\langle 1/p^n \mid n \in \mathbb{N} \rangle \subseteq M_p$.
3. Set $M := M_p \times M_p$. Then M is atomic and contains any element of the form $(a/p^n, b/p^m)$ for $a, b, m, n \in \mathbb{N}_0$.
4. For every $n \in \mathbb{N}$ with $\gcd(p, n) = 1$, the polynomial $X^n + Y^n + X^n Y^n$ is irreducible in $F[X, Y]$.

An Answer to G'

Theorem (Coykendall-G: 2018)

For each field F of finite characteristic, there exists an atomic monoid M such that $F[M]$ is not atomic.

Sketch of Proof:

1. Let $\{p_n\}$ be the strictly increasing sequence of primes. Set $p := \mathrm{char}(F)$, and consider the Puiseux monoid
$$M_p := \left\langle \frac{1}{p^n p_n} \;\middle|\; p_n \neq p \right\rangle.$$
Note that M_2 is the Gram's monoid.
2. The monoid M_p is atomic, but $\langle 1/p^n \mid n \in \mathbb{N} \rangle \subseteq M_p$.
3. Set $M := M_p \times M_p$. Then M is atomic and contains any element of the form $(a/p^n, b/p^m)$ for $a, b, m, n \in \mathbb{N}_0$.
4. For every $n \in \mathbb{N}$ with $\gcd(p, n) = 1$, the polynomial $X^n + Y^n + X^n Y^n$ is irreducible in $F[X, Y]$.

An Answer to G'

Theorem (Coykendall-G: 2018)

For each field F of finite characteristic, there exists an atomic monoid M such that $F[M]$ is not atomic.

Sketch of Proof:

1. Let $\{p_n\}$ be the strictly increasing sequence of primes. Set $p := \text{char}(F)$, and consider the Puiseux monoid
$$M_p := \left\langle \frac{1}{p^n p_n} \;\middle|\; p_n \neq p \right\rangle.$$
Note that M_2 is the Gram's monoid.
2. The monoid M_p is atomic, but $\langle 1/p^n \mid n \in \mathbb{N} \rangle \subseteq M_p$.
3. Set $M := M_p \times M_p$. Then M is atomic and contains any element of the form $(a/p^n, b/p^m)$ for $a, b, m, n \in \mathbb{N}_0$.
4. For every $n \in \mathbb{N}$ with $\gcd(p, n) = 1$, the polynomial $X^n + Y^n + X^n Y^n$ is irreducible in $F[X, Y]$.

An Answer to G'

Theorem (Coykendall-G: 2018)

For each field F of finite characteristic, there exists an atomic monoid M such that $F[M]$ is not atomic.

Sketch of Proof:

1. Let $\{p_n\}$ be the strictly increasing sequence of primes. Set $p := \text{char}(F)$, and consider the Puiseux monoid
$$M_p := \Big\langle \frac{1}{p^n p_n} \;\Big|\; p_n \neq p \Big\rangle.$$
Note that M_2 is the Gram's monoid.
2. The monoid M_p is atomic, but $\langle 1/p^n \mid n \in \mathbb{N} \rangle \subseteq M_p$.
3. Set $M := M_p \times M_p$. Then M is atomic and contains any element of the form $(a/p^n, b/p^m)$ for $a, b, m, n \in \mathbb{N}_0$.
4. For every $n \in \mathbb{N}$ with $\gcd(p, n) = 1$, the polynomial $X^n + Y^n + X^n Y^n$ is irreducible in $F[X, Y]$.

An Answer to G'

Theorem (Coykendall-G: 2018)

For each field F of finite characteristic, there exists an atomic monoid M such that $F[M]$ is not atomic.

Sketch of Proof:

1. Let $\{p_n\}$ be the strictly increasing sequence of primes. Set $p := \text{char}(F)$, and consider the Puiseux monoid
$$M_p := \left\langle \frac{1}{p^n p_n} \;\middle|\; p_n \neq p \right\rangle.$$
Note that M_2 is the Gram's monoid.
2. The monoid M_p is atomic, but $\langle 1/p^n \mid n \in \mathbb{N} \rangle \subseteq M_p$.
3. Set $M := M_p \times M_p$. Then M is atomic and contains any element of the form $(a/p^n, b/p^m)$ for $a, b, m, n \in \mathbb{N}_0$.
4. For every $n \in \mathbb{N}$ with $\gcd(p, n) = 1$, the polynomial $X^n + Y^n + X^n Y^n$ is irreducible in $F[X, Y]$.

An Answer to G'

Theorem (Coykendall-G: 2018)

For each field F of finite characteristic, there exists an atomic monoid M such that $F[M]$ is not atomic.

Sketch of Proof:

1. Let $\{p_n\}$ be the strictly increasing sequence of primes. Set $p := \mathrm{char}(F)$, and consider the Puiseux monoid
$$M_p := \Big\langle \frac{1}{p^n p_n} \ \Big| \ p_n \neq p \Big\rangle.$$
Note that M_2 is the Gram's monoid.
2. The monoid M_p is atomic, but $\langle 1/p^n \mid n \in \mathbb{N} \rangle \subseteq M_p$.
3. Set $M := M_p \times M_p$. Then M is atomic and contains any element of the form $(a/p^n, b/p^m)$ for $a, b, m, n \in \mathbb{N}_0$.
4. For every $n \in \mathbb{N}$ with $\gcd(p, n) = 1$, the polynomial $X^n + Y^n + X^n Y^n$ is irreducible in $F[X, Y]$.

An Answer to G' (continuation)

Theorem (Coykendall-G: 2018)

For each field F of finite characteristic, there exists an atomic monoid M such that $F[M]$ is not atomic.

Sketch of Proof (continuation):

4. For every $n \in \mathbb{N}$ with $\gcd(p, n) = 1$, the polynomial $X^n + Y^n + X^n Y^n$ is irreducible in $F[X, Y]$.
5. Consider the element $f := X + Y + XY \in F[X, Y; M]$.
6. Each nonunit divisor of f has the form $\left(X^{\frac{1}{p^k}} + Y^{\frac{1}{p^k}} + X^{\frac{1}{p^k}} Y^{\frac{1}{p^k}}\right)^t$ for some $k \in \mathbb{N}_0$ and $t \in \mathbb{N}$.
7. f is not irreducible as $f = \left(X^{\frac{1}{p}} + Y^{\frac{1}{p}} + X^{\frac{1}{p}} Y^{\frac{1}{p}}\right)^p$.
8. any factor g of f in a potential decomposition into irreducibles in $F[X, Y; M]$ must be of the form $X^{\frac{1}{p^k}} + Y^{\frac{1}{p^k}} + X^{\frac{1}{p^k}} Y^{\frac{1}{p^k}}$.
9. Then $g = \left(X^{\frac{1}{p^{k+1}}} + Y^{\frac{1}{p^{k+1}}} + X^{\frac{1}{p^{k+1}}} Y^{\frac{1}{p^{k+1}}}\right)^p$, which contradicts that g is irreducible. □

An Answer to G' (continuation)

Theorem (Coykendall-G: 2018)

For each field F of finite characteristic, there exists an atomic monoid M such that $F[M]$ is not atomic.

Sketch of Proof (continuation):

4. For every $n \in \mathbb{N}$ with $\gcd(p, n) = 1$, the polynomial $X^n + Y^n + X^n Y^n$ is irreducible in $F[X, Y]$.
5. Consider the element $f := X + Y + XY \in F[X, Y; M]$.
6. Each nonunit divisor of f has the form $\left(X^{\frac{1}{p^k}} + Y^{\frac{1}{p^k}} + X^{\frac{1}{p^k}} Y^{\frac{1}{p^k}}\right)^t$ for some $k \in \mathbb{N}_0$ and $t \in \mathbb{N}$.
7. f is not irreducible as $f = \left(X^{\frac{1}{p}} + Y^{\frac{1}{p}} + X^{\frac{1}{p}} Y^{\frac{1}{p}}\right)^p$.
8. any factor g of f in a potential decomposition into irreducibles in $F[X, Y; M]$ must be of the form $X^{\frac{1}{p^k}} + Y^{\frac{1}{p^k}} + X^{\frac{1}{p^k}} Y^{\frac{1}{p^k}}$.
9. Then $g = \left(X^{\frac{1}{p^{k+1}}} + Y^{\frac{1}{p^{k+1}}} + X^{\frac{1}{p^{k+1}}} Y^{\frac{1}{p^{k+1}}}\right)^p$, which contradicts that g is irreducible. □

An Answer to G' (continuation)

Theorem (Coykendall-G: 2018)

For each field F of finite characteristic, there exists an atomic monoid M such that $F[M]$ is not atomic.

Sketch of Proof (continuation):

4. For every $n \in \mathbb{N}$ with $\gcd(p, n) = 1$, the polynomial $X^n + Y^n + X^n Y^n$ is irreducible in $F[X, Y]$.
5. Consider the element $f := X + Y + XY \in F[X, Y; M]$.
6. Each nonunit divisor of f has the form $\left(X^{\frac{1}{p^k}} + Y^{\frac{1}{p^k}} + X^{\frac{1}{p^k}} Y^{\frac{1}{p^k}}\right)^t$ for some $k \in \mathbb{N}_0$ and $t \in \mathbb{N}$.
7. f is not irreducible as $f = \left(X^{\frac{1}{p}} + Y^{\frac{1}{p}} + X^{\frac{1}{p}} Y^{\frac{1}{p}}\right)^p$.
8. any factor g of f in a potential decomposition into irreducibles in $F[X, Y; M]$ must be of the form $X^{\frac{1}{p^k}} + Y^{\frac{1}{p^k}} + X^{\frac{1}{p^k}} Y^{\frac{1}{p^k}}$.
9. Then $g = \left(X^{\frac{1}{p^{k+1}}} + Y^{\frac{1}{p^{k+1}}} + X^{\frac{1}{p^{k+1}}} Y^{\frac{1}{p^{k+1}}}\right)^p$, which contradicts that g is irreducible. □

An Answer to G' (continuation)

Theorem (Coykendall-G: 2018)

For each field F of finite characteristic, there exists an atomic monoid M such that $F[M]$ is not atomic.

Sketch of Proof (continuation):

4. For every $n \in \mathbb{N}$ with $\gcd(p, n) = 1$, the polynomial $X^n + Y^n + X^n Y^n$ is irreducible in $F[X, Y]$.
5. Consider the element $f := X + Y + XY \in F[X, Y; M]$.
6. Each nonunit divisor of f has the form $\big(X^{\frac{1}{p^k}} + Y^{\frac{1}{p^k}} + X^{\frac{1}{p^k}} Y^{\frac{1}{p^k}}\big)^t$ for some $k \in \mathbb{N}_0$ and $t \in \mathbb{N}$.
7. f is not irreducible as $f = \big(X^{\frac{1}{p}} + Y^{\frac{1}{p}} + X^{\frac{1}{p}} Y^{\frac{1}{p}}\big)^p$.
8. any factor g of f in a potential decomposition into irreducibles in $F[X, Y; M]$ must be of the form $X^{\frac{1}{p^k}} + Y^{\frac{1}{p^k}} + X^{\frac{1}{p^k}} Y^{\frac{1}{p^k}}$.
9. Then $g = \big(X^{\frac{1}{p^{k+1}}} + Y^{\frac{1}{p^{k+1}}} + X^{\frac{1}{p^{k+1}}} Y^{\frac{1}{p^{k+1}}}\big)^p$, which contradicts that g is irreducible. □

An Answer to G' (continuation)

Theorem (Coykendall-G: 2018)

For each field F of finite characteristic, there exists an atomic monoid M such that $F[M]$ is not atomic.

Sketch of Proof (continuation):

4. For every $n \in \mathbb{N}$ with $\gcd(p, n) = 1$, the polynomial $X^n + Y^n + X^n Y^n$ is irreducible in $F[X, Y]$.
5. Consider the element $f := X + Y + XY \in F[X, Y; M]$.
6. Each nonunit divisor of f has the form $\left(X^{\frac{1}{p^k}} + Y^{\frac{1}{p^k}} + X^{\frac{1}{p^k}} Y^{\frac{1}{p^k}}\right)^t$ for some $k \in \mathbb{N}_0$ and $t \in \mathbb{N}$.
7. f is not irreducible as $f = \left(X^{\frac{1}{p}} + Y^{\frac{1}{p}} + X^{\frac{1}{p}} Y^{\frac{1}{p}}\right)^p$.
8. any factor g of f in a potential decomposition into irreducibles in $F[X, Y; M]$ must be of the form $X^{\frac{1}{p^k}} + Y^{\frac{1}{p^k}} + X^{\frac{1}{p^k}} Y^{\frac{1}{p^k}}$.
9. Then $g = \left(X^{\frac{1}{p^{k+1}}} + Y^{\frac{1}{p^{k+1}}} + X^{\frac{1}{p^{k+1}}} Y^{\frac{1}{p^{k+1}}}\right)^p$, which contradicts that g is irreducible. □

An Answer to G' (continuation)

Theorem (Coykendall-G: 2018)

For each field F of finite characteristic, there exists an atomic monoid M such that $F[M]$ is not atomic.

Sketch of Proof (continuation):

4. For every $n \in \mathbb{N}$ with $\gcd(p, n) = 1$, the polynomial $X^n + Y^n + X^n Y^n$ is irreducible in $F[X, Y]$.
5. Consider the element $f := X + Y + XY \in F[X, Y; M]$.
6. Each nonunit divisor of f has the form $\big(X^{\frac{1}{p^k}} + Y^{\frac{1}{p^k}} + X^{\frac{1}{p^k}} Y^{\frac{1}{p^k}}\big)^t$ for some $k \in \mathbb{N}_0$ and $t \in \mathbb{N}$.
7. f is not irreducible as $f = \big(X^{\frac{1}{p}} + Y^{\frac{1}{p}} + X^{\frac{1}{p}} Y^{\frac{1}{p}}\big)^p$.
8. any factor g of f in a potential decomposition into irreducibles in $F[X, Y; M]$ must be of the form $X^{\frac{1}{p^k}} + Y^{\frac{1}{p^k}} + X^{\frac{1}{p^k}} Y^{\frac{1}{p^k}}$.
9. Then $g = \big(X^{\frac{1}{p^{k+1}}} + Y^{\frac{1}{p^{k+1}}} + X^{\frac{1}{p^{k+1}}} Y^{\frac{1}{p^{k+1}}}\big)^p$, which contradicts that g is irreducible. $\square$

Is G' True When Restricted to Rank-1 Monoids?

Remark. Recall that the monoids used to answer Question G' were $M_p \times M_p$. As $M_p \times M_p \cong \alpha M_p \oplus \beta M_p$, where $\alpha, \beta \in \mathbb{R}_{>0} \setminus \mathbb{Q}$ are linearly independent over $\mathbb{Q}$, the rank of $M_p \times M_p$ is 2 for any prime p.

Question

Is G' true when restricted to monoids of rank 1, i.e., Puiseux monoids?

YES:

Theorem (Coykendall-G: 2018)

There exists an atomic Puiseux monoid M such that the monoid algebra $\mathbb{Z}_2[M]$ is not atomic.

Is G' True When Restricted to Rank-1 Monoids?

Remark. Recall that the monoids used to answer Question G' were $M_p \times M_p$. As $M_p \times M_p \cong \alpha M_p \oplus \beta M_p$, where $\alpha, \beta \in \mathbb{R}_{>0} \setminus \mathbb{Q}$ are linearly independent over $\mathbb{Q}$, the rank of $M_p \times M_p$ is 2 for any prime p.

Question

Is G' true when restricted to monoids of rank 1*, i.e., Puiseux monoids?*

YES:

Theorem (Coykendall-G: 2018)

There exists an atomic Puiseux monoid M such that the monoid algebra $\mathbb{Z}_2[M]$ is not atomic.

Is G' True When Restricted to Rank-1 Monoids?

Remark. Recall that the monoids used to answer Question G' were $M_p \times M_p$. As $M_p \times M_p \cong \alpha M_p \oplus \beta M_p$, where $\alpha, \beta \in \mathbb{R}_{>0} \setminus \mathbb{Q}$ are linearly independent over $\mathbb{Q}$, the rank of $M_p \times M_p$ is 2 for any prime p.

Question

Is G' true when restricted to monoids of rank 1*, i.e., Puiseux monoids?*

YES:

Theorem (Coykendall-G: 2018)

There exists an atomic Puiseux monoid M such that the monoid algebra $\mathbb{Z}_2[M]$ is not atomic.

Is G' True When Restricted to Rank-1 Monoids?

Remark. Recall that the monoids used to answer Question G' were $M_p \times M_p$. As $M_p \times M_p \cong \alpha M_p \oplus \beta M_p$, where $\alpha, \beta \in \mathbb{R}_{>0} \setminus \mathbb{Q}$ are linearly independent over $\mathbb{Q}$, the rank of $M_p \times M_p$ is 2 for any prime p.

Question

Is G' true when restricted to monoids of rank 1, i.e., Puiseux monoids?

YES:

Theorem (Coykendall-G: 2018)

There exists an atomic Puiseux monoid M such that the monoid algebra $\mathbb{Z}_2[M]$ is not atomic.

Is G' True When Restricted to Rank-1 Monoids? (continuation)

Theorem (Coykendall-G: 2018)

There exists an atomic Puiseux monoid M such that the monoid algebra $\mathbb{Z}_2[M]$ is not atomic.

Sketch of Proof:

1. Let $\{\ell_n\}$ be a strictly increasing sequence of positive integers satisfying that $3^{\ell_n - \ell_{n-1}} > 2^{n+1}$.
2. Consider the Puiseux monoid

$$M := \left\langle \frac{2^n 3^{\ell_n} - 1}{2^{2n} 3^{\ell_n}}, \frac{2^n 3^{\ell_n} + 1}{2^{2n} 3^{\ell_n}} \;\middle|\; n \in \mathbb{N} \right\rangle.$$

3. The monoid M is atomic, but $\langle 1/2^n \mid n \in \mathbb{N} \rangle \subset M$.
4. For each $n \in \mathbb{N}$, the polynomial $x^{2 \cdot 3^n} + x^{3^n} + 1$ is irreducible in $\mathbb{Z}_2[x]$.
5. The element $x^2 + x + 1$ cannot be expressed as a product of irreducibles in $\mathbb{Z}_2[M]$, and to argue this we can proceed as in the (sketch of) proof of the previous theorem. □

Is G' True When Restricted to Rank-1 Monoids? (continuation)

Theorem (Coykendall-G: 2018)

There exists an atomic Puiseux monoid M such that the monoid algebra $\mathbb{Z}_2[M]$ is not atomic.

Sketch of Proof:

1. Let $\{\ell_n\}$ be a strictly increasing sequence of positive integers satisfying that $3^{\ell_n - \ell_{n-1}} > 2^{n+1}$.
2. Consider the Puiseux monoid

$$M := \left\langle \frac{2^n 3^{\ell_n} - 1}{2^{2n} 3^{\ell_n}}, \frac{2^n 3^{\ell_n} + 1}{2^{2n} 3^{\ell_n}} \mid n \in \mathbb{N} \right\rangle.$$

3. The monoid M is atomic, but $\langle 1/2^n \mid n \in \mathbb{N} \rangle \subset M$.
4. For each $n \in \mathbb{N}$, the polynomial $x^{2 \cdot 3^n} + x^{3^n} + 1$ is irreducible in $\mathbb{Z}_2[x]$.
5. The element $x^2 + x + 1$ cannot be expressed as a product of irreducibles in $\mathbb{Z}_2[M]$, and to argue this we can proceed as in the (sketch of) proof of the previous theorem. □

Is G' True When Restricted to Rank-1 Monoids? (continuation)

Theorem (Coykendall-G: 2018)

There exists an atomic Puiseux monoid M such that the monoid algebra $\mathbb{Z}_2[M]$ is not atomic.

Sketch of Proof:

1. Let $\{\ell_n\}$ be a strictly increasing sequence of positive integers satisfying that $3^{\ell_n - \ell_{n-1}} > 2^{n+1}$.
2. Consider the Puiseux monoid

$$M := \left\langle \frac{2^n 3^{\ell_n} - 1}{2^{2n} 3^{\ell_n}}, \frac{2^n 3^{\ell_n} + 1}{2^{2n} 3^{\ell_n}} \mid n \in \mathbb{N} \right\rangle.$$

3. The monoid M is atomic, but $\langle 1/2^n \mid n \in \mathbb{N} \rangle \subset M$.
4. For each $n \in \mathbb{N}$, the polynomial $x^{2 \cdot 3^n} + x^{3^n} + 1$ is irreducible in $\mathbb{Z}_2[x]$.
5. The element $x^2 + x + 1$ cannot be expressed as a product of irreducibles in $\mathbb{Z}_2[M]$, and to argue this we can proceed as in the (sketch of) proof of the previous theorem. $\square$

Is G' True When Restricted to Rank-1 Monoids? (continuation)

Theorem (Coykendall-G: 2018)

There exists an atomic Puiseux monoid M such that the monoid algebra $\mathbb{Z}_2[M]$ is not atomic.

Sketch of Proof:

1. Let $\{\ell_n\}$ be a strictly increasing sequence of positive integers satisfying that $3^{\ell_n - \ell_{n-1}} > 2^{n+1}$.
2. Consider the Puiseux monoid
$$M := \Big\langle \frac{2^n 3^{\ell_n} - 1}{2^{2n} 3^{\ell_n}}, \frac{2^n 3^{\ell_n} + 1}{2^{2n} 3^{\ell_n}} \;\Big|\; n \in \mathbb{N} \Big\rangle.$$
3. The monoid M is atomic, but $\langle 1/2^n \mid n \in \mathbb{N} \rangle \subset M$.
4. For each $n \in \mathbb{N}$, the polynomial $x^{2 \cdot 3^n} + x^{3^n} + 1$ is irreducible in $\mathbb{Z}_2[x]$.
5. The element $x^2 + x + 1$ cannot be expressed as a product of irreducibles in $\mathbb{Z}_2[M]$, and to argue this we can proceed as in the (sketch of) proof of the previous theorem. □

Is G' True When Restricted to Rank-1 Monoids? (continuation)

Theorem (Coykendall-G: 2018)

There exists an atomic Puiseux monoid M such that the monoid algebra $\mathbb{Z}_2[M]$ is not atomic.

Sketch of Proof:

1. Let $\{\ell_n\}$ be a strictly increasing sequence of positive integers satisfying that $3^{\ell_n - \ell_{n-1}} > 2^{n+1}$.
2. Consider the Puiseux monoid
$$M := \Big\langle \frac{2^n 3^{\ell_n} - 1}{2^{2n} 3^{\ell_n}}, \frac{2^n 3^{\ell_n} + 1}{2^{2n} 3^{\ell_n}} \ \Big|\ n \in \mathbb{N} \Big\rangle.$$
3. The monoid M is atomic, but $\langle 1/2^n \mid n \in \mathbb{N} \rangle \subset M$.
4. For each $n \in \mathbb{N}$, the polynomial $x^{2 \cdot 3^n} + x^{3^n} + 1$ is irreducible in $\mathbb{Z}_2[x]$.
5. The element $x^2 + x + 1$ cannot be expressed as a product of irreducibles in $\mathbb{Z}_2[M]$, and to argue this we can proceed as in the (sketch of) proof of the previous theorem. □

Is G' True When Restricted to Rank-1 Monoids? (continuation)

Theorem (Coykendall-G: 2018)

There exists an atomic Puiseux monoid M such that the monoid algebra $\mathbb{Z}_2[M]$ is not atomic.

Sketch of Proof:

1. Let $\{\ell_n\}$ be a strictly increasing sequence of positive integers satisfying that $3^{\ell_n - \ell_{n-1}} > 2^{n+1}$.
2. Consider the Puiseux monoid

$$M := \Big\langle \frac{2^n 3^{\ell_n} - 1}{2^{2n} 3^{\ell_n}}, \frac{2^n 3^{\ell_n} + 1}{2^{2n} 3^{\ell_n}} \ \Big| \ n \in \mathbb{N} \Big\rangle.$$

3. The monoid M is atomic, but $\langle 1/2^n \mid n \in \mathbb{N} \rangle \subset M$.
4. For each $n \in \mathbb{N}$, the polynomial $x^{2 \cdot 3^n} + x^{3^n} + 1$ is irreducible in $\mathbb{Z}_2[x]$.
5. The element $x^2 + x + 1$ cannot be expressed as a product of irreducibles in $\mathbb{Z}_2[M]$, and to argue this we can proceed as in the (sketch of) proof of the previous theorem. □

Is G' True When Restricted to Rank-1 Monoids? (continuation)

Theorem (Coykendall-G: 2018)

There exists an atomic Puiseux monoid M such that the monoid algebra $\mathbb{Z}_2[M]$ is not atomic.

Sketch of Proof:

1. Let $\{\ell_n\}$ be a strictly increasing sequence of positive integers satisfying that $3^{\ell_n - \ell_{n-1}} > 2^{n+1}$.
2. Consider the Puiseux monoid
$$M := \Big\langle \frac{2^n 3^{\ell_n} - 1}{2^{2n} 3^{\ell_n}}, \frac{2^n 3^{\ell_n} + 1}{2^{2n} 3^{\ell_n}} \;\Big|\; n \in \mathbb{N} \Big\rangle.$$
3. The monoid M is atomic, but $\langle 1/2^n \mid n \in \mathbb{N} \rangle \subset M$.
4. For each $n \in \mathbb{N}$, the polynomial $x^{2 \cdot 3^n} + x^{3^n} + 1$ is irreducible in $\mathbb{Z}_2[x]$.
5. The element $x^2 + x + 1$ cannot be expressed as a product of irreducibles in $\mathbb{Z}_2[M]$, and to argue this we can proceed as in the (sketch of) proof of the previous theorem. □

Related Open Questions

Question

For each prime p, can we find a pair (M, F), where M is a rank-1 atomic monoid and F is a field of characteristic p such that $F[M]$ is not atomic?

Question

Can we find a pair (M, F), where M is an atomic monoid and F is a field of characteristic 0 such that $F[M]$ is not atomic?

Related Open Questions

Question

For each prime p, can we find a pair (M, F), where M is a rank-1 atomic monoid and F is a field of characteristic p such that $F[M]$ is not atomic?

Question

Can we find a pair (M, F), where M is an atomic monoid and F is a field of characteristic 0 such that $F[M]$ is not atomic?

References

- D. D. Anderson, D. F. Anderson, and M. Zafrullah: *Factorizations in integral domains*, J. Pure Appl. Algebra **69** (1990) 1–19.
- R. Gilmer: *Commutative Semigroup Rings*, Chicago Lectures in Mathematics, The University of Chicago Press, London, 1984.
- R. Gilmer and T. Parker: *Divisibility properties of semigroup rings*, Mich. Math. J. **21** (1974) 65–86.
- A. Grams: *Atomic domains and the ascending chain condition for principal ideals*, Math. Proc. Cambridge Philos. Soc. **75** (1974) 321–329.
- M. Roitman: *Polynomial extensions of atomic domains*, J. Pure Appl. Algebra **87** (1993) 187–199.

End of Presentation

THANK YOU!