Cyclic rational semirings

A comparison between the factorization invariants of numerical monoids and Puiseux monoids

Marly Gotti

March 22, 2019

AMS Sectional Meeting,
University of Hawaii at Manoa

Table of contents

General comparison between numerical and Puiseux monoids

Definition

If N is a submonoid of $(\mathbb{N}_0, +)$ such that $\mathbb{N}_0 \setminus N$ is finite, then N is called a numerical monoid.

Definition

A submonoid of $(\mathbb{Q}_{\geq 0}, +)$ is called a Puiseux monoid.

Note: Every numerical monoid is a Puiseux monoid.

Definition

If N is a submonoid of $(\mathbb{N}_0, +)$ such that $\mathbb{N}_0 \setminus N$ is finite, then N is called a numerical monoid.

Definition

A submonoid of $(\mathbb{Q}_{\geq 0}, +)$ is called a Puiseux monoid.

Note: Every numerical monoid is a Puiseux monoid.

Table 1: Factorization invariants of numerical monoids vs. Puiseux monoids

Let N be a numerical monoid	Let M be a Puiseux monoid
System of sets of lengths	
Sets of lengths are arithmetic multiprogressions. Also, for $L \subseteq \mathbb{N}_{\geq 2}$, there is a numerical monoid N and $x \in N$ with $\mathsf{L}(x) = L$.	Sets of lengths can have arbitrary behavior. There exists a Puiseux monoid M such that for any $L \subseteq \mathbb{N}_{\geq 2}$, there is an $x \in M$ with $\mathsf{L}(x) = L$.
Elasticity	
$\rho(N) = \frac{\max \mathcal{A}(N)}{\min \mathcal{A}(N)}$ is always finite and accepted. Moreover, N is fully elastic if and only if N is isomorphic to $(\mathbb{N}_0, +)$.	If M is atomic, then $\rho(M) = \infty$ if 0 is a limit point of $\mathcal{A}(M)$ and $\rho(M) = \frac{\sup \mathcal{A}(M)}{\inf \mathcal{A}(M)}$ otherwise. Moreover, $\rho(M)$ is accepted if and only if $\mathcal{A}(M)$ has a min and a max in $\mathbb{Q}$.
Catenary degree	
$\mathsf{c}(N) \leq \frac{\mathcal{F}(N) + \max \mathcal{A}(N)}{\min \mathcal{A}(N)} + 1$.	No known general results.

Table 1: Factorization invariants of numerical monoids vs. Puiseux monoids

Let N be a numerical monoid	Let M be a Puiseux monoid
System of sets of lengths	
Sets of lengths are arithmetic multiprogressions. Also, for $L \subseteq \mathbb{N}_{\ge 2}$, there is a numerical monoid N and $x \in N$ with $\mathsf{L}(x) = L$.	Sets of lengths can have arbitrary behavior. There exists a Puiseux monoid M such that for any $L \subseteq \mathbb{N}_{\ge 2}$, there is an $x \in M$ with $\mathsf{L}(x) = L$.
Elasticity	
$\rho(N) = \frac{\max \mathcal{A}(N)}{\min \mathcal{A}(N)}$ is always finite and accepted. Moreover, N is fully elastic if and only if N is isomorphic to $(\mathbb{N}_0, +)$.	If M is atomic, then $\rho(M) = \infty$ if 0 is a limit point of $\mathcal{A}(M)$ and $\rho(M) = \frac{\sup \mathcal{A}(M)}{\inf \mathcal{A}(M)}$ otherwise. Moreover, $\rho(M)$ is accepted if and only if $\mathcal{A}(M)$ has a min and a max in $\mathbb{Q}$.
Catenary degree	
$\mathsf{c}(N) \le \frac{\mathcal{F}(N) + \max \mathcal{A}(N)}{\min \mathcal{A}(N)} + 1.$	No known general results.

Table 1: Factorization invariants of numerical monoids vs. Puiseux monoids

Let N be a numerical monoid	**Let M be a Puiseux monoid**
System of sets of lengths	
Sets of lengths are arithmetic multiprogressions. Also, for $L \subseteq \mathbb{N}_{\geq 2}$, there is a numerical monoid N and $x \in N$ with $\mathsf{L}(x) = L$.	Sets of lengths can have arbitrary behavior. There exists a Puiseux monoid M such that for any $L \subseteq \mathbb{N}_{\geq 2}$, there is an $x \in M$ with $\mathsf{L}(x) = L$.
Elasticity	
$\rho(N) = \frac{\max \mathcal{A}(N)}{\min \mathcal{A}(N)}$ is always finite and accepted. Moreover, N is fully elastic if and only if N is isomorphic to $(\mathbb{N}_0, +)$.	If M is atomic, then $\rho(M) = \infty$ if 0 is a limit point of $\mathcal{A}(M)$ and $\rho(M) = \frac{\sup \mathcal{A}(M)}{\inf \mathcal{A}(M)}$ otherwise. Moreover, $\rho(M)$ is accepted if and only if $\mathcal{A}(M)$ has a min and a max in $\mathbb{Q}$.
Catenary degree	
$\mathsf{c}(N) \leq \frac{\mathcal{F}(N) + \max \mathcal{A}(N)}{\min \mathcal{A}(N)} + 1$.	No known general results.

Table 1: Factorization invariants of numerical monoids vs. Puiseux monoids

Let N be a numerical monoid	**Let M be a Puiseux monoid**
System of sets of lengths	
Sets of lengths are arithmetic multiprogressions. Also, for $L \subseteq \mathbb{N}_{\geq 2}$, there is a numerical monoid N and $x \in N$ with $\mathsf{L}(x) = L$.	Sets of lengths can have arbitrary behavior. There exists a Puiseux monoid M such that for any $L \subseteq \mathbb{N}_{\geq 2}$, there is an $x \in M$ with $\mathsf{L}(x) = L$.
Elasticity	
$\rho(N) = \frac{\max \mathcal{A}(N)}{\min \mathcal{A}(N)}$ is always finite and accepted. Moreover, N is fully elastic if and only if N is isomorphic to $(\mathbb{N}_0, +)$.	If M is atomic, then $\rho(M) = \infty$ if 0 is a limit point of $\mathcal{A}(M)$ and $\rho(M) = \frac{\sup \mathcal{A}(M)}{\inf \mathcal{A}(M)}$ otherwise. Moreover, $\rho(M)$ is accepted if and only if $\mathcal{A}(M)$ has a min and a max in $\mathbb{Q}$.
Catenary degree	
$\mathsf{c}(N) \leq \frac{\mathcal{F}(N) + \max \mathcal{A}(N)}{\min \mathcal{A}(N)} + 1.$	No known general results.

Table 1: Factorization invariants of numerical monoids vs. Puiseux monoids

Let N be a numerical monoid	Let M be a Puiseux monoid
Is it finitely generated?	
Always.	M is finitely generated if and only if M is isomorphic to a numerical monoid.
Is it atomic?	
Always.	$\langle 1/2^n \mid n \in \mathbb{N}_0 \rangle$ is not atomic. M is atomic if 0 is not a limit point of M.
Is it a BF-monoid (BFM)?	
Always.	M can be atomic and not a BFM.
Is it an FF-monoid (FFM)?	
Always.	M can be a BFM and not an FFM.

Table 1: Factorization invariants of numerical monoids vs. Puiseux monoids

Let N be a numerical monoid	**Let M be a Puiseux monoid**
Is it finitely generated?	
Always.	M is finitely generated if and only if M is isomorphic to a numerical monoid.
Is it atomic?	
Always.	$\langle 1/2^n \mid n \in \mathbb{N}_0 \rangle$ is not atomic. M is atomic if 0 is not a limit point of M.
Is it a BF-monoid (BFM)?	
Always.	M can be atomic and not a BFM.
Is it an FF-monoid (FFM)?	
Always.	M can be a BFM and not an FFM.

Table 1: Factorization invariants of numerical monoids vs. Puiseux monoids

Let N be a numerical monoid	Let M be a Puiseux monoid
Is it finitely generated?	
Always.	M is finitely generated if and only if M is isomorphic to a numerical monoid.
Is it atomic?	
Always.	$\langle 1/2^n \mid n \in \mathbb{N}_0 \rangle$ is not atomic. M is atomic if 0 is not a limit point of M.
Is it a BF-monoid (BFM)?	
Always.	M can be atomic and not a BFM.
Is it an FF-monoid (FFM)?	
Always.	M can be a BFM and not an FFM.

Table 1: Factorization invariants of numerical monoids vs. Puiseux monoids

Let N be a numerical monoid	Let M be a Puiseux monoid
Is it finitely generated?	
Always.	M is finitely generated if and only if M is isomorphic to a numerical monoid.
Is it atomic?	
Always.	$\langle 1/2^n \mid n \in \mathbb{N}_0 \rangle$ is not atomic. M is atomic if 0 is not a limit point of M.
Is it a BF-monoid (BFM)?	
Always.	M can be atomic and not a BFM.
Is it an FF-monoid (FFM)?	
Always.	M can be a BFM and not an FFM.

Table 1: Factorization invariants of numerical monoids vs. Puiseux monoids

Let N be a numerical monoid	**Let M be a Puiseux monoid**
Is it finitely generated?	
Always.	M is finitely generated if and only if M is isomorphic to a numerical monoid.
Is it atomic?	
Always.	$\langle 1/2^n \mid n \in \mathbb{N}_0 \rangle$ is not atomic. M is atomic if 0 is not a limit point of M.
Is it a BF-monoid (BFM)?	
Always.	M can be atomic and not a BFM.
Is it an FF-monoid (FFM)?	
Always.	M can be a BFM and not an FFM.

Definitions and background

Definitions

Let M be a reduced commutative cancellative monoid.

- $M^\bullet$ denotes the set $M \setminus \{0\}$.
- We write $M = \langle S \rangle$ when M is generated by a set S. We say that M is finitely generated if it can be generated by a finite set.
- An element $a \in M^\bullet$ is called an atom provided that for each pair of elements $x, y \in M$ such that $a = x + y$ either $x = 0$ or $y = 0$. The set of atoms of M is denoted by $\mathcal{A}(M)$. If $\mathcal{A}(M)$ generates M, then M is called atomic.
- The factorization monoid of M is the free commutative monoid on $\mathcal{A}(M)$ and is denoted by $\mathsf{Z}(M)$. The elements of $\mathsf{Z}(M)$ are called factorizations.
- If $z = a_1 + \cdots + a_n$ is a factorization of M for some $a_1, \ldots, a_n \in \mathcal{A}(M)$, then n is called the length of z and is denoted by $|z|$.

Definitions

Let M be a reduced commutative cancellative monoid.

- $M^\bullet$ denotes the set $M \setminus \{0\}$.
- We write $M = \langle S \rangle$ when M is generated by a set S. We say that M is finitely generated if it can be generated by a finite set.
- An element $a \in M^\bullet$ is called an atom provided that for each pair of elements $x, y \in M$ such that $a = x + y$ either $x = 0$ or $y = 0$. The set of atoms of M is denoted by $\mathcal{A}(M)$. If $\mathcal{A}(M)$ generates M, then M is called atomic.
- The factorization monoid of M is the free commutative monoid on $\mathcal{A}(M)$ and is denoted by $\mathsf{Z}(M)$. The elements of $\mathsf{Z}(M)$ are called factorizations.
- If $z = a_1 + \cdots + a_n$ is a factorization of M for some $a_1, \dots, a_n \in \mathcal{A}(M)$, then n is called the length of z and is denoted by $|z|$.

Definitions

Let M be a reduced commutative cancellative monoid.

- $M^\bullet$ denotes the set $M \setminus \{0\}$.
- We write $M = \langle S \rangle$ when M is generated by a set S. We say that M is finitely generated if it can be generated by a finite set.
- An element $a \in M^\bullet$ is called an atom provided that for each pair of elements $x, y \in M$ such that $a = x + y$ either $x = 0$ or $y = 0$. The set of atoms of M is denoted by $\mathcal{A}(M)$. If $\mathcal{A}(M)$ generates M, then M is called atomic.
- The factorization monoid of M is the free commutative monoid on $\mathcal{A}(M)$ and is denoted by $\mathsf{Z}(M)$. The elements of $\mathsf{Z}(M)$ are called factorizations.
- If $z = a_1 + \cdots + a_n$ is a factorization of M for some $a_1, \ldots, a_n \in \mathcal{A}(M)$, then n is called the length of z and is denoted by $|z|$.

Definitions

Let M be a reduced commutative cancellative monoid.

- $M^\bullet$ denotes the set $M \setminus \{0\}$.
- We write $M = \langle S \rangle$ when M is generated by a set S. We say that M is finitely generated if it can be generated by a finite set.
- An element $a \in M^\bullet$ is called an atom provided that for each pair of elements $x, y \in M$ such that $a = x + y$ either $x = 0$ or $y = 0$. The set of atoms of M is denoted by $\mathcal{A}(M)$. If $\mathcal{A}(M)$ generates M, then M is called atomic.
- The factorization monoid of M is the free commutative monoid on $\mathcal{A}(M)$ and is denoted by $\mathsf{Z}(M)$. The elements of $\mathsf{Z}(M)$ are called factorizations.
- If $z = a_1 + \cdots + a_n$ is a factorization of M for some $a_1, \dots, a_n \in \mathcal{A}(M)$, then n is called the length of z and is denoted by $|z|$.

Definitions

Let M be a reduced commutative cancellative monoid.

- $M^\bullet$ denotes the set $M \setminus \{0\}$.
- We write $M = \langle S \rangle$ when M is generated by a set S. We say that M is finitely generated if it can be generated by a finite set.
- An element $a \in M^\bullet$ is called an atom provided that for each pair of elements $x, y \in M$ such that $a = x + y$ either $x = 0$ or $y = 0$. The set of atoms of M is denoted by $\mathcal{A}(M)$. If $\mathcal{A}(M)$ generates M, then M is called atomic.
- The factorization monoid of M is the free commutative monoid on $\mathcal{A}(M)$ and is denoted by $\mathsf{Z}(M)$. The elements of $\mathsf{Z}(M)$ are called factorizations.
- If $z = a_1 + \cdots + a_n$ is a factorization of M for some $a_1, \ldots, a_n \in \mathcal{A}(M)$, then n is called the length of z and is denoted by $|z|$.

Definitions (Cont.)

- The unique monoid homomorphism $\phi\colon \mathsf{Z}(M) \to M$ satisfying $\phi(a) = a$ for all $a \in \mathcal{A}(M)$ is called the factorization homomorphism of M.
- For each $x \in M$, the set

$$\mathsf{Z}(x) := \phi^{-1}(x) \subseteq \mathsf{Z}(M)$$

  is called the set of factorizations of x and the set

$$\mathsf{L}(x) := \{|z| : z \in \mathsf{Z}(x)\}$$

  is called the set of lengths of x.
- If $\mathsf{L}(x)$ is a finite set for all $x \in M$, then M is called a bounded factorization monoid or a BF-monoid.
- If M is an atomic monoid, the elasticity of an element $x \in M^\bullet$, denoted by $\rho(x)$, is defined as

$$\rho(x) = \frac{\sup \mathsf{L}(x)}{\inf \mathsf{L}(x)}.$$

Definitions (Cont.)

- The unique monoid homomorphism $\phi\colon \mathsf{Z}(M) \to M$ satisfying $\phi(a) = a$ for all $a \in \mathcal{A}(M)$ is called the factorization homomorphism of M.
- For each $x \in M$, the set

  $$\mathsf{Z}(x) := \phi^{-1}(x) \subseteq \mathsf{Z}(M)$$

  is called the set of factorizations of x and the set

  $$\mathsf{L}(x) := \{|z| : z \in \mathsf{Z}(x)\}$$

  is called the set of lengths of x.
- If $\mathsf{L}(x)$ is a finite set for all $x \in M$, then M is called a bounded factorization monoid or a BF-monoid.
- If M is an atomic monoid, the elasticity of an element $x \in M^{\bullet}$, denoted by $\rho(x)$, is defined as

  $$\rho(x) = \frac{\sup \mathsf{L}(x)}{\inf \mathsf{L}(x)}.$$

Definitions (Cont.)

- The unique monoid homomorphism $\phi\colon \mathsf{Z}(M) \to M$ satisfying $\phi(a) = a$ for all $a \in \mathcal{A}(M)$ is called the factorization homomorphism of M.
- For each $x \in M$, the set

$$\mathsf{Z}(x) := \phi^{-1}(x) \subseteq \mathsf{Z}(M)$$

  is called the set of factorizations of x and the set

$$\mathsf{L}(x) := \{|z| : z \in \mathsf{Z}(x)\}$$

  is called the set of lengths of x.
- If $\mathsf{L}(x)$ is a finite set for all $x \in M$, then M is called a bounded factorization monoid or a BF-monoid.
- If M is an atomic monoid, the elasticity of an element $x \in M^\bullet$, denoted by $\rho(x)$, is defined as

$$\rho(x) = \frac{\sup \mathsf{L}(x)}{\inf \mathsf{L}(x)}.$$

Definitions (Cont.)

- The unique monoid homomorphism $\phi\colon \mathsf{Z}(M) \to M$ satisfying $\phi(a) = a$ for all $a \in \mathcal{A}(M)$ is called the factorization homomorphism of M.
- For each $x \in M$, the set

$$\mathsf{Z}(x) := \phi^{-1}(x) \subseteq \mathsf{Z}(M)$$

  is called the set of factorizations of x and the set

$$\mathsf{L}(x) := \{|z| : z \in \mathsf{Z}(x)\}$$

  is called the set of lengths of x.
- If $\mathsf{L}(x)$ is a finite set for all $x \in M$, then M is called a bounded factorization monoid or a BF-monoid.
- If M is an atomic monoid, the elasticity of an element $x \in M^{\bullet}$, denoted by $\rho(x)$, is defined as

$$\rho(x) = \frac{\sup \mathsf{L}(x)}{\inf \mathsf{L}(x)}.$$

Introduction to cyclic rational semirings

Cyclic rational semirings

Definition

For $r \in \mathbb{Q}_{>0}$, we call cyclic rational semiring to the Puiseux monoid S_r generated by the nonnegative powers of r, i.e.,

$$S_r = \langle r^n \mid n \in \mathbb{N}_0 \rangle.$$

Theorem [Gotti-G., 2017]

For $r \in \mathbb{Q}_{>0}$, let S_r be the cyclic rational semiring generated by r. Then the following statements hold.

- If $\mathsf{d}(r) = 1$, then S_r is atomic with $\mathcal{A}(S_r) = \{1\}$.
- If $\mathsf{d}(r) > 1$ and $\mathsf{n}(r) = 1$, then S_r contains no atoms.
- If $\mathsf{d}(r) > 1$ and $\mathsf{n}(r) > 1$, then S_r is atomic with $\mathcal{A}(S_r) = \{r^n \mid n \in \mathbb{N}_0\}$.

Definition

For $r \in \mathbb{Q}_{>0}$, we call cyclic rational semiring to the Puiseux monoid S_r generated by the nonnegative powers of r, i.e.,

$$S_r = \langle r^n \mid n \in \mathbb{N}_0 \rangle.$$

Theorem [Gotti-G., 2017]

For $r \in \mathbb{Q}_{>0}$, let S_r be the cyclic rational semiring generated by r. Then the following statements hold.

- If $\mathrm{d}(r) = 1$, then S_r is atomic with $\mathcal{A}(S_r) = \{1\}$.
- If $\mathrm{d}(r) > 1$ and $\mathrm{n}(r) = 1$, then S_r contains no atoms.
- If $\mathrm{d}(r) > 1$ and $\mathrm{n}(r) > 1$, then S_r is atomic with $\mathcal{A}(S_r) = \{r^n \mid n \in \mathbb{N}_0\}$.

Definition

For $r \in \mathbb{Q}_{>0}$, we call cyclic rational semiring to the Puiseux monoid S_r generated by the nonnegative powers of r, i.e.,

$$S_r = \langle r^n \mid n \in \mathbb{N}_0 \rangle.$$

Theorem [Gotti-G., 2017]

For $r \in \mathbb{Q}_{>0}$, let S_r be the cyclic rational semiring generated by r. Then the following statements hold.

- If $\mathrm{d}(r) = 1$, then S_r is atomic with $\mathcal{A}(S_r) = \{1\}$.
- If $\mathrm{d}(r) > 1$ and $\mathrm{n}(r) = 1$, then S_r contains no atoms.
- If $\mathrm{d}(r) > 1$ and $\mathrm{n}(r) > 1$, then S_r is atomic with $\mathcal{A}(S_r) = \{r^n \mid n \in \mathbb{N}_0\}$.

Cyclic rational semirings

Definition

For $r \in \mathbb{Q}_{>0}$, we call cyclic rational semiring to the Puiseux monoid S_r generated by the nonnegative powers of r, i.e.,

$$S_r = \langle r^n \mid n \in \mathbb{N}_0 \rangle.$$

Theorem [Gotti-G., 2017]

For $r \in \mathbb{Q}_{>0}$, let S_r be the cyclic rational semiring generated by r. Then the following statements hold.

- If $\mathrm{d}(r) = 1$, then S_r is atomic with $\mathcal{A}(S_r) = \{1\}$.
- If $\mathrm{d}(r) > 1$ and $\mathrm{n}(r) = 1$, then S_r contains no atoms.
- If $\mathrm{d}(r) > 1$ and $\mathrm{n}(r) > 1$, then S_r is atomic with $\mathcal{A}(S_r) = \{r^n \mid n \in \mathbb{N}_0\}$.

Definition

For $r \in \mathbb{Q}_{>0}$, we call cyclic rational semiring to the Puiseux monoid S_r generated by the nonnegative powers of r, i.e.,

$$S_r = \langle r^n \mid n \in \mathbb{N}_0 \rangle.$$

Theorem [Gotti-G., 2017]

For $r \in \mathbb{Q}_{>0}$, let S_r be the cyclic rational semiring generated by r. Then the following statements hold.

- If $\mathrm{d}(r) = 1$, then S_r is atomic with $\mathcal{A}(S_r) = \{1\}$.
- If $\mathrm{d}(r) > 1$ and $\mathrm{n}(r) = 1$, then S_r contains no atoms.
- If $\mathrm{d}(r) > 1$ and $\mathrm{n}(r) > 1$, then S_r is atomic with $\mathcal{A}(S_r) = \{r^n \mid n \in \mathbb{N}_0\}$.

Set of lengths, delta set, and catenary degree

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic, and for $x \in S_r^\bullet$ let $z = \sum_{i=0}^{N} \alpha_i r^i \in \mathsf{Z}(x)$, where $N \in \mathbb{N}$ and $\alpha_0, \dots, \alpha_N \in \mathbb{N}_0$.

Lemma [Chapman-Gotti-G.]

If $r \in (0,1) \cap \mathbb{Q}$, then the next statements hold.

1. There is exactly one factorization in $\mathsf{Z}(x)$ of minimum length and $\sup \mathsf{L}(x) \in \{1, \infty\}$.
2. $|\mathsf{Z}(x)| = 1$ if and only if $|\mathsf{L}(x)| = 1$.

Lemma [Chapman-Gotti-G.]

If $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$, then the following statements hold.

1. There exist exactly one factorization in $\mathsf{Z}(x)$ of minimum length and exactly one factorization of maximum length.
2. $|\mathsf{Z}(x)| = 1$ if and only if $|\mathsf{L}(x)| = 1$.

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic, and for $x \in S_r^\bullet$ let $z = \sum_{i=0}^{N} \alpha_i r^i \in \mathsf{Z}(x)$, where $N \in \mathbb{N}$ and $\alpha_0, \ldots, \alpha_N \in \mathbb{N}_0$.

Lemma [Chapman-Gotti-G.]

If $r \in (0,1) \cap \mathbb{Q}$, then the next statements hold.

1. There is exactly one factorization in $\mathsf{Z}(x)$ of minimum length and $\sup \mathsf{L}(x) \in \{1, \infty\}$.
2. $|\mathsf{Z}(x)| = 1$ if and only if $|\mathsf{L}(x)| = 1$.

Lemma [Chapman-Gotti-G.]

If $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$, then the following statements hold.

1. There exist exactly one factorization in $\mathsf{Z}(x)$ of minimum length and exactly one factorization of maximum length.
2. $|\mathsf{Z}(x)| = 1$ if and only if $|\mathsf{L}(x)| = 1$.

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic, and for $x \in S_r^\bullet$ let $z = \sum_{i=0}^N \alpha_i r^i \in \mathsf{Z}(x)$, where $N \in \mathbb{N}$ and $\alpha_0, \ldots, \alpha_N \in \mathbb{N}_0$.

Lemma [Chapman-Gotti-G.]

If $r \in (0,1) \cap \mathbb{Q}$, then the next statements hold.

1. There is exactly one factorization in $\mathsf{Z}(x)$ of minimum length and $\sup \mathsf{L}(x) \in \{1, \infty\}$.
2. $|\mathsf{Z}(x)| = 1$ if and only if $|\mathsf{L}(x)| = 1$.

Lemma [Chapman-Gotti-G.]

If $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$, then the following statements hold.

1. There exist exactly one factorization in $\mathsf{Z}(x)$ of minimum length and exactly one factorization of maximum length.
2. $|\mathsf{Z}(x)| = 1$ if and only if $|\mathsf{L}(x)| = 1$.

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic, and for $x \in S_r^\bullet$ let $z = \sum_{i=0}^{N} \alpha_i r^i \in \mathsf{Z}(x)$, where $N \in \mathbb{N}$ and $\alpha_0, \dots, \alpha_N \in \mathbb{N}_0$.

Lemma [Chapman-Gotti-G.]

If $r \in (0,1) \cap \mathbb{Q}$, then the next statements hold.

1. There is exactly one factorization in $\mathsf{Z}(x)$ of minimum length and $\sup \mathsf{L}(x) \in \{1, \infty\}$.
2. $|\mathsf{Z}(x)| = 1$ if and only if $|\mathsf{L}(x)| = 1$.

Lemma [Chapman-Gotti-G.]

If $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$, then the following statements hold.

1. There exist exactly one factorization in $\mathsf{Z}(x)$ of minimum length and exactly one factorization of maximum length.
2. $|\mathsf{Z}(x)| = 1$ if and only if $|\mathsf{L}(x)| = 1$.

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic, and for $x \in S_r^\bullet$ let $z = \sum_{i=0}^{N} \alpha_i r^i \in \mathsf{Z}(x)$, where $N \in \mathbb{N}$ and $\alpha_0, \dots, \alpha_N \in \mathbb{N}_0$.

Lemma [Chapman-Gotti-G.]

If $r \in (0,1) \cap \mathbb{Q}$, then the next statements hold.

1. There is exactly one factorization in $\mathsf{Z}(x)$ of minimum length and $\sup \mathsf{L}(x) \in \{1, \infty\}$.
2. $|\mathsf{Z}(x)| = 1$ if and only if $|\mathsf{L}(x)| = 1$.

Lemma [Chapman-Gotti-G.]

If $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$, then the following statements hold.

1. There exist exactly one factorization in $\mathsf{Z}(x)$ of minimum length and exactly one factorization of maximum length.
2. $|\mathsf{Z}(x)| = 1$ if and only if $|\mathsf{L}(x)| = 1$.

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic, and for $x \in S_r^\bullet$ let $z = \sum_{i=0}^{N} \alpha_i r^i \in \mathsf{Z}(x)$, where $N \in \mathbb{N}$ and $\alpha_0, \dots, \alpha_N \in \mathbb{N}_0$.

Lemma [Chapman-Gotti-G.]

If $r \in (0,1) \cap \mathbb{Q}$, then the next statements hold.

1. There is exactly one factorization in $\mathsf{Z}(x)$ of minimum length and $\sup \mathsf{L}(x) \in \{1, \infty\}$.
2. $|\mathsf{Z}(x)| = 1$ if and only if $|\mathsf{L}(x)| = 1$.

Lemma [Chapman-Gotti-G.]

If $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$, then the following statements hold.

1. There exist exactly one factorization in $\mathsf{Z}(x)$ of minimum length and exactly one factorization of maximum length.
2. $|\mathsf{Z}(x)| = 1$ if and only if $|\mathsf{L}(x)| = 1$.

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic, and for $x \in S_r^\bullet$ let $z = \sum_{i=0}^{N} \alpha_i r^i \in \mathsf{Z}(x)$, where $N \in \mathbb{N}$ and $\alpha_0, \ldots, \alpha_N \in \mathbb{N}_0$.

Lemma [Chapman-Gotti-G.]

If $r \in (0,1) \cap \mathbb{Q}$, then the next statements hold.

1. There is exactly one factorization in $\mathsf{Z}(x)$ of minimum length and $\sup \mathsf{L}(x) \in \{1, \infty\}$.
2. $|\mathsf{Z}(x)| = 1$ if and only if $|\mathsf{L}(x)| = 1$.

Lemma [Chapman-Gotti-G.]

If $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$, then the following statements hold.

1. There exist exactly one factorization in $\mathsf{Z}(x)$ of minimum length and exactly one factorization of maximum length.
2. $|\mathsf{Z}(x)| = 1$ if and only if $|\mathsf{L}(x)| = 1$.

Main result on the set of lengths

Theorem [Chapman-Gotti-G.]

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic.

1. If $r < 1$, then for each $x \in S_r$ with $|\mathsf{Z}(x)| > 1$,
$$\mathsf{L}(x) = \left\{ \min \mathsf{L}(x) + k\big(\mathsf{d}(r) - \mathsf{n}(r)\big) \mid k \in \mathbb{N}_0 \right\}.$$
2. If $r \in \mathbb{N}$, then $|\mathsf{Z}(x)| = |\mathsf{L}(x)| = 1$ for all $x \in S_r^\bullet$.
3. If $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$, then for each $x \in S_r$ with $|\mathsf{Z}(x)| > 1$,
$$\mathsf{L}(x) = \left\{ \min \mathsf{L}(x) + k\big(\mathsf{n}(r) - \mathsf{d}(r)\big) \;\middle|\; 0 \le k \le \frac{\max \mathsf{L}(x) - \min \mathsf{L}(x)}{\mathsf{n}(r) - \mathsf{d}(r)} \right\}.$$

Thus, the set $\mathsf{L}(x)$ is an arithmetic progression with difference $|\mathsf{n}(r) - \mathsf{d}(r)|$.

Theorem [Chapman-Gotti-G.]

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic.

1. If $r < 1$, then for each $x \in S_r$ with $|\mathsf{Z}(x)| > 1$,

$$\mathsf{L}(x) = \left\{ \min \mathsf{L}(x) + k\big(\mathsf{d}(r) - \mathsf{n}(r)\big) \mid k \in \mathbb{N}_0 \right\}.$$

2. If $r \in \mathbb{N}$, then $|\mathsf{Z}(x)| = |\mathsf{L}(x)| = 1$ for all $x \in S_r^\bullet$.
3. If $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$, then for each $x \in S_r$ with $|\mathsf{Z}(x)| > 1$,

$$\mathsf{L}(x) = \left\{ \min \mathsf{L}(x) + k\big(\mathsf{n}(r) - \mathsf{d}(r)\big) \;\middle|\; 0 \le k \le \frac{\max \mathsf{L}(x) - \min \mathsf{L}(x)}{\mathsf{n}(r) - \mathsf{d}(r)} \right\}.$$

Thus, the set $\mathsf{L}(x)$ is an arithmetic progression with difference $|\mathsf{n}(r) - \mathsf{d}(r)|$.

Theorem [Chapman-Gotti-G.]

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic.

1. If $r < 1$, then for each $x \in S_r$ with $|\mathsf{Z}(x)| > 1$,

$$\mathsf{L}(x) = \big\{ \min \mathsf{L}(x) + k\big(\mathsf{d}(r) - \mathsf{n}(r)\big) \mid k \in \mathbb{N}_0 \big\}.$$

2. If $r \in \mathbb{N}$, then $|\mathsf{Z}(x)| = |\mathsf{L}(x)| = 1$ for all $x \in S_r^{\bullet}$.
3. If $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$, then for each $x \in S_r$ with $|\mathsf{Z}(x)| > 1$,

$$\mathsf{L}(x) = \left\{ \min \mathsf{L}(x) + k\big(\mathsf{n}(r) - \mathsf{d}(r)\big) \;\middle|\; 0 \le k \le \frac{\max \mathsf{L}(x) - \min \mathsf{L}(x)}{\mathsf{n}(r) - \mathsf{d}(r)} \right\}.$$

Thus, the set $\mathsf{L}(x)$ is an arithmetic progression with difference $|\mathsf{n}(r) - \mathsf{d}(r)|$.

Theorem [Chapman-Gotti-G.]

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic.

1. If $r < 1$, then for each $x \in S_r$ with $|\mathsf{Z}(x)| > 1$,
$$\mathsf{L}(x) = \big\{ \min \mathsf{L}(x) + k\big(\mathsf{d}(r) - \mathsf{n}(r)\big) \mid k \in \mathbb{N}_0 \big\}.$$
2. If $r \in \mathbb{N}$, then $|\mathsf{Z}(x)| = |\mathsf{L}(x)| = 1$ for all $x \in S_r^\bullet$.
3. If $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$, then for each $x \in S_r$ with $|\mathsf{Z}(x)| > 1$,
$$\mathsf{L}(x) = \left\{ \min \mathsf{L}(x) + k\big(\mathsf{n}(r) - \mathsf{d}(r)\big) \;\middle|\; 0 \le k \le \frac{\max \mathsf{L}(x) - \min \mathsf{L}(x)}{\mathsf{n}(r) - \mathsf{d}(r)} \right\}.$$

Thus, the set $\mathsf{L}(x)$ is an arithmetic progression with difference $|\mathsf{n}(r) - \mathsf{d}(r)|$.

Theorem [Chapman-Gotti-G.]

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic.

1. If $r < 1$, then for each $x \in S_r$ with $|\mathsf{Z}(x)| > 1$,
$$\mathsf{L}(x) = \left\{ \min \mathsf{L}(x) + k\big(\mathsf{d}(r) - \mathsf{n}(r)\big) \mid k \in \mathbb{N}_0 \right\}.$$
2. If $r \in \mathbb{N}$, then $|\mathsf{Z}(x)| = |\mathsf{L}(x)| = 1$ for all $x \in S_r^{\bullet}$.
3. If $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$, then for each $x \in S_r$ with $|\mathsf{Z}(x)| > 1$,
$$\mathsf{L}(x) = \left\{ \min \mathsf{L}(x) + k\big(\mathsf{n}(r) - \mathsf{d}(r)\big) \;\middle|\; 0 \le k \le \frac{\max \mathsf{L}(x) - \min \mathsf{L}(x)}{\mathsf{n}(r) - \mathsf{d}(r)} \right\}.$$

Thus, the set $\mathsf{L}(x)$ is an arithmetic progression with difference $|\mathsf{n}(r) - \mathsf{d}(r)|$.

Elasticity

Conjecture on the missing cases

Corollary [Chapman-Gotti-G.]

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic. Then, either $\rho(S_r) = 1$ or $\rho(S_r) = \infty$.

For an atomic monoid M the set $R(M) = \{\rho(x) \mid x \in M\}$ is called the set of elasticities of M. The monoid M is called fully elastic if $R(M) = \mathbb{Q} \cap [1, \rho(M)]$ when $\infty \notin R(M)$ and $R(M) \setminus \{\infty\} = \mathbb{Q} \cap [1, \infty)$ when $\infty \in R(M)$.

Proposition [Chapman-Gotti-G.]

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic.

1. If $r < 1$, then $R(S_r) = \{1, \infty\}$ and, therefore, S_r is not fully elastic.
2. If $r \in \mathbb{N}$, then $R(S_r) = \{1\}$ and, therefore, S_r is fully elastic.
3. If $r \notin \mathbb{N}$ and $\mathrm{n}(r) = \mathrm{d}(r) + 1$, then S_r is fully elastic, in which case $R(S_r) = \mathbb{Q}_{\geq 1}$.

Proposition [Chapman-Gotti-G.]

The set of elasticities of S_r is dense in $\mathbb{R}_{\geq 1}$ if and only if $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$.

Conjecture

For $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$ such that $\mathrm{n}(r) > \mathrm{d}(r) + 1$, the monoid S_r is fully elastic.

Conjecture on the missing cases

Corollary [Chapman-Gotti-G.]

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic. Then, either $\rho(S_r) = 1$ or $\rho(S_r) = \infty$.

For an atomic monoid M the set $R(M) = \{\rho(x) \mid x \in M\}$ is called the set of elasticities of M. The monoid M is called fully elastic if $R(M) = \mathbb{Q} \cap [1, \rho(M)]$ when $\infty \notin R(M)$ and $R(M) \setminus \{\infty\} = \mathbb{Q} \cap [1, \infty)$ when $\infty \in R(M)$.

Proposition [Chapman-Gotti-G.]

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic.

1. If $r < 1$, then $R(S_r) = \{1, \infty\}$ and, therefore, S_r is not fully elastic.
2. If $r \in \mathbb{N}$, then $R(S_r) = \{1\}$ and, therefore, S_r is fully elastic.
3. If $r \notin \mathbb{N}$ and $\mathrm{n}(r) = \mathrm{d}(r) + 1$, then S_r is fully elastic, in which case $R(S_r) = \mathbb{Q}_{\geq 1}$.

Proposition [Chapman-Gotti-G.]

The set of elasticities of S_r is dense in $\mathbb{R}_{\geq 1}$ if and only if $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$.

Conjecture

For $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$ such that $\mathrm{n}(r) > \mathrm{d}(r) + 1$, the monoid S_r is fully elastic.

Conjecture on the missing cases

Corollary [Chapman-Gotti-G.]

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic. Then, either $\rho(S_r) = 1$ or $\rho(S_r) = \infty$.

For an atomic monoid M the set $R(M) = \{\rho(x) \mid x \in M\}$ is called the set of elasticities of M. The monoid M is called fully elastic if $R(M) = \mathbb{Q} \cap [1, \rho(M)]$ when $\infty \notin R(M)$ and $R(M) \setminus \{\infty\} = \mathbb{Q} \cap [1, \infty)$ when $\infty \in R(M)$.

Proposition [Chapman-Gotti-G.]

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic.

1. If $r < 1$, then $R(S_r) = \{1, \infty\}$ and, therefore, S_r is not fully elastic.
2. If $r \in \mathbb{N}$, then $R(S_r) = \{1\}$ and, therefore, S_r is fully elastic.
3. If $r \notin \mathbb{N}$ and $\mathrm{n}(r) = \mathrm{d}(r) + 1$, then S_r is fully elastic, in which case $R(S_r) = \mathbb{Q}_{\geq 1}$.

Proposition [Chapman-Gotti-G.]

The set of elasticities of S_r is dense in $\mathbb{R}_{\geq 1}$ if and only if $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$.

Conjecture

For $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$ such that $\mathrm{n}(r) > \mathrm{d}(r) + 1$, the monoid S_r is fully elastic.

Conjecture on the missing cases

Corollary [Chapman-Gotti-G.]

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic. Then, either $\rho(S_r) = 1$ or $\rho(S_r) = \infty$.

For an atomic monoid M the set $R(M) = \{\rho(x) \mid x \in M\}$ is called the set of elasticities of M. The monoid M is called fully elastic if $R(M) = \mathbb{Q} \cap [1, \rho(M)]$ when $\infty \notin R(M)$ and $R(M) \setminus \{\infty\} = \mathbb{Q} \cap [1, \infty)$ when $\infty \in R(M)$.

Proposition [Chapman-Gotti-G.]

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic.

1. If $r < 1$, then $R(S_r) = \{1, \infty\}$ and, therefore, S_r is not fully elastic.
2. If $r \in \mathbb{N}$, then $R(S_r) = \{1\}$ and, therefore, S_r is fully elastic.
3. If $r \notin \mathbb{N}$ and $\mathrm{n}(r) = \mathrm{d}(r) + 1$, then S_r is fully elastic, in which case $R(S_r) = \mathbb{Q}_{\geq 1}$.

Proposition [Chapman-Gotti-G.]

The set of elasticities of S_r is dense in $\mathbb{R}_{\geq 1}$ if and only if $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$.

Conjecture

For $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$ such that $\mathrm{n}(r) > \mathrm{d}(r) + 1$, the monoid S_r is fully elastic.

Conjecture on the missing cases

Corollary [Chapman-Gotti-G.]

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic. Then, either $\rho(S_r) = 1$ or $\rho(S_r) = \infty$.

For an atomic monoid M the set $R(M) = \{\rho(x) \mid x \in M\}$ is called the set of elasticities of M. The monoid M is called fully elastic if $R(M) = \mathbb{Q} \cap [1, \rho(M)]$ when $\infty \notin R(M)$ and $R(M) \setminus \{\infty\} = \mathbb{Q} \cap [1, \infty)$ when $\infty \in R(M)$.

Proposition [Chapman-Gotti-G.]

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic.

1. If $r < 1$, then $R(S_r) = \{1, \infty\}$ and, therefore, S_r is not fully elastic.
2. If $r \in \mathbb{N}$, then $R(S_r) = \{1\}$ and, therefore, S_r is fully elastic.
3. If $r \notin \mathbb{N}$ and $\mathrm{n}(r) = \mathrm{d}(r) + 1$, then S_r is fully elastic, in which case $R(S_r) = \mathbb{Q}_{\geq 1}$.

Proposition [Chapman-Gotti-G.]

The set of elasticities of S_r is dense in $\mathbb{R}_{\geq 1}$ if and only if $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$.

Conjecture

For $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$ such that $\mathrm{n}(r) > \mathrm{d}(r) + 1$, the monoid S_r is fully elastic.

Conjecture on the missing cases

Corollary [Chapman-Gotti-G.]

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic. Then, either $\rho(S_r) = 1$ or $\rho(S_r) = \infty$.

For an atomic monoid M the set $R(M) = \{\rho(x) \mid x \in M\}$ is called the set of elasticities of M. The monoid M is called fully elastic if $R(M) = \mathbb{Q} \cap [1, \rho(M)]$ when $\infty \notin R(M)$ and $R(M) \setminus \{\infty\} = \mathbb{Q} \cap [1, \infty)$ when $\infty \in R(M)$.

Proposition [Chapman-Gotti-G.]

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic.

1. If $r < 1$, then $R(S_r) = \{1, \infty\}$ and, therefore, S_r is not fully elastic.
2. If $r \in \mathbb{N}$, then $R(S_r) = \{1\}$ and, therefore, S_r is fully elastic.
3. If $r \notin \mathbb{N}$ and $\mathrm{n}(r) = \mathrm{d}(r) + 1$, then S_r is fully elastic, in which case $R(S_r) = \mathbb{Q}_{\geq 1}$.

Proposition [Chapman-Gotti-G.]

The set of elasticities of S_r is dense in $\mathbb{R}_{\geq 1}$ if and only if $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$.

Conjecture

For $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$ such that $\mathrm{n}(r) > \mathrm{d}(r) + 1$, the monoid S_r is fully elastic.

Conjecture on the missing cases

Corollary [Chapman-Gotti-G.]

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic. Then, either $\rho(S_r) = 1$ or $\rho(S_r) = \infty$.

For an atomic monoid M the set $R(M) = \{\rho(x) \mid x \in M\}$ is called the set of elasticities of M. The monoid M is called fully elastic if $R(M) = \mathbb{Q} \cap [1, \rho(M)]$ when $\infty \notin R(M)$ and $R(M) \setminus \{\infty\} = \mathbb{Q} \cap [1, \infty)$ when $\infty \in R(M)$.

Proposition [Chapman-Gotti-G.]

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic.

1. If $r < 1$, then $R(S_r) = \{1, \infty\}$ and, therefore, S_r is not fully elastic.
2. If $r \in \mathbb{N}$, then $R(S_r) = \{1\}$ and, therefore, S_r is fully elastic.
3. If $r \notin \mathbb{N}$ and $\mathrm{n}(r) = \mathrm{d}(r) + 1$, then S_r is fully elastic, in which case $R(S_r) = \mathbb{Q}_{\geq 1}$.

Proposition [Chapman-Gotti-G.]

The set of elasticities of S_r is dense in $\mathbb{R}_{\geq 1}$ if and only if $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$.

Conjecture

For $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$ such that $\mathrm{n}(r) > \mathrm{d}(r) + 1$, the monoid S_r is fully elastic.

Conjecture on the missing cases

Corollary [Chapman-Gotti-G.]

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic. Then, either $\rho(S_r) = 1$ or $\rho(S_r) = \infty$.

For an atomic monoid M the set $R(M) = \{\rho(x) \mid x \in M\}$ is called the set of elasticities of M. The monoid M is called fully elastic if $R(M) = \mathbb{Q} \cap [1, \rho(M)]$ when $\infty \notin R(M)$ and $R(M) \setminus \{\infty\} = \mathbb{Q} \cap [1, \infty)$ when $\infty \in R(M)$.

Proposition [Chapman-Gotti-G.]

Take $r \in \mathbb{Q}_{>0}$ such that S_r is atomic.

1. If $r < 1$, then $R(S_r) = \{1, \infty\}$ and, therefore, S_r is not fully elastic.
2. If $r \in \mathbb{N}$, then $R(S_r) = \{1\}$ and, therefore, S_r is fully elastic.
3. If $r \notin \mathbb{N}$ and $\mathrm{n}(r) = \mathrm{d}(r) + 1$, then S_r is fully elastic, in which case $R(S_r) = \mathbb{Q}_{\geq 1}$.

Proposition [Chapman-Gotti-G.]

The set of elasticities of S_r is dense in $\mathbb{R}_{\geq 1}$ if and only if $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$.

Conjecture

For $r \in \mathbb{Q}_{>1} \setminus \mathbb{N}$ such that $\mathrm{n}(r) > \mathrm{d}(r) + 1$, the monoid S_r is fully elastic.

Comparing two classes of nicely generated monoids

Table 2: Factorization Invariants Comparison

Numerical monoids of the form $N = \langle n, n+d, \dots, n+kd \rangle$	Puiseux monoids of the form $S_r = \langle r^n \mid n \in \mathbb{N}_0 \rangle$
System of sets of lengths	
Sets of lengths in N are arithmetic progressions. By these results, $\Delta(N) = \{d\}$.	Sets of lengths in S_r are arithmetic progressions. A a consequence, $\Delta(S_r) = \{\lvert \mathsf{n}(r) - \mathsf{d}(r) \rvert\}$.
Elasticity	
$\rho(N) = \frac{n+dk}{n}$ is accepted. It is fully elastic only when $N = \mathbb{N}_0$.	If S_r is atomic, then $\rho(S_r) \in \{1, \infty\}$. Moreover, $\rho(M)$ is accepted if and only if $r < 1$ or $r \in \mathbb{N}$. S_r is fully elastic when $\mathsf{n}(r) = \mathsf{d}(r) + 1$.
Catenary degree	
$\mathsf{c}(N) = \lceil \frac{n}{k} \rceil + d$.	If S_r is atomic, then $\mathsf{c}(S_r) = \lvert \mathsf{n}(r) - \mathsf{d}(r) \rvert$.
Omega primality	
$\omega(N) < \infty$.	If S_r is atomic and $r \in \mathbb{Q} \cap (0,1)$, then $\omega(S_r) = \infty$.

Table 2: Factorization Invariants Comparison

Numerical monoids of the form $N = \langle n, n+d, \dots, n+kd \rangle$	Puiseux monoids of the form $S_r = \langle r^n \mid n \in \mathbb{N}_0 \rangle$
System of sets of lengths	
Sets of lengths in N are arithmetic progressions. By these results, $\Delta(N) = \{d\}$.	Sets of lengths in S_r are arithmetic progressions. A a consequence, $\Delta(S_r) = \{\lvert \mathsf{n}(r) - \mathsf{d}(r) \rvert\}$.
Elasticity	
$\rho(N) = \frac{n+dk}{n}$ is accepted. It is fully elastic only when $N = \mathbb{N}_0$.	If S_r is atomic, then $\rho(S_r) \in \{1, \infty\}$. Moreover, $\rho(M)$ is accepted if and only if $r < 1$ or $r \in \mathbb{N}$. S_r is fully elastic when $\mathsf{n}(r) = \mathsf{d}(r) + 1$.
Catenary degree	
$\mathsf{c}(N) = \lceil \frac{n}{k} \rceil + d$.	If S_r is atomic, then $\mathsf{c}(S_r) = \lvert \mathsf{n}(r) - \mathsf{d}(r) \rvert$.
Omega primality	
$\omega(N) < \infty$.	If S_r is atomic and $r \in \mathbb{Q} \cap (0, 1)$, then $\omega(S_r) = \infty$.

Table 2: Factorization Invariants Comparison

Numerical monoids of the form $N = \langle n, n+d, \ldots, n+kd \rangle$	Puiseux monoids of the form $S_r = \langle r^n \mid n \in \mathbb{N}_0 \rangle$
System of sets of lengths	
Sets of lengths in N are arithmetic progressions. By these results, $\Delta(N) = \{d\}$.	Sets of lengths in S_r are arithmetic progressions. A a consequence, $\Delta(S_r) = \{\lvert \mathsf{n}(r) - \mathsf{d}(r) \rvert\}$.
Elasticity	
$\rho(N) = \frac{n+dk}{n}$ is accepted. It is fully elastic only when $N = \mathbb{N}_0$.	If S_r is atomic, then $\rho(S_r) \in \{1, \infty\}$. Moreover, $\rho(M)$ is accepted if and only if $r < 1$ or $r \in \mathbb{N}$. S_r is fully elastic when $\mathsf{n}(r) = \mathsf{d}(r) + 1$.
Catenary degree	
$\mathsf{c}(N) = \lceil \frac{n}{k} \rceil + d$.	If S_r is atomic, then $\mathsf{c}(S_r) = \lvert \mathsf{n}(r) - \mathsf{d}(r) \rvert$.
Omega primality	
$\omega(N) < \infty$.	If S_r is atomic and $r \in \mathbb{Q} \cap (0,1)$, then $\omega(S_r) = \infty$.

Table 2: Factorization Invariants Comparison

Numerical monoids of the form $N = \langle n, n+d, \dots, n+kd \rangle$	Puiseux monoids of the form $S_r = \langle r^n \mid n \in \mathbb{N}_0 \rangle$
System of sets of lengths	
Sets of lengths in N are arithmetic progressions. By these results, $\Delta(N) = \{d\}$.	Sets of lengths in S_r are arithmetic progressions. A a consequence, $\Delta(S_r) = \{\lvert n(r) - d(r) \rvert\}$.
Elasticity	
$\rho(N) = \frac{n+dk}{n}$ is accepted. It is fully elastic only when $N = \mathbb{N}_0$.	If S_r is atomic, then $\rho(S_r) \in \{1, \infty\}$. Moreover, $\rho(M)$ is accepted if and only if $r < 1$ or $r \in \mathbb{N}$. S_r is fully elastic when $n(r) = d(r) + 1$.
Catenary degree	
$c(N) = \lceil \frac{n}{k} \rceil + d$.	If S_r is atomic, then $c(S_r) = \lvert n(r) - d(r) \rvert$.
Omega primality	
$\omega(N) < \infty$.	If S_r is atomic and $r \in \mathbb{Q} \cap (0, 1)$, then $\omega(S_r) = \infty$.

Table 2: Factorization Invariants Comparison

Numerical monoids of the form $\mathbf{N = \langle n, n+d, \dots, n+kd \rangle}$	**Puiseux monoids of the form $\mathbf{S_r = \langle r^n \mid n \in \mathbb{N}_0 \rangle}$**
System of sets of lengths	
Sets of lengths in N are arithmetic progressions. By these results, $\Delta(N) = \{d\}$.	Sets of lengths in S_r are arithmetic progressions. A a consequence, $\Delta(S_r) = \{\lvert \mathrm{n}(r) - \mathrm{d}(r) \rvert\}$.
Elasticity	
$\rho(N) = \frac{n+dk}{n}$ is accepted. It is fully elastic only when $N = \mathbb{N}_0$.	If S_r is atomic, then $\rho(S_r) \in \{1, \infty\}$. Moreover, $\rho(M)$ is accepted if and only if $r < 1$ or $r \in \mathbb{N}$. S_r is fully elastic when $\mathrm{n}(r) = \mathrm{d}(r) + 1$.
Catenary degree	
$\mathrm{c}(N) = \lceil \frac{n}{k} \rceil + d$.	If S_r is atomic, then $\mathrm{c}(S_r) = \lvert \mathrm{n}(r) - \mathrm{d}(r) \rvert$.
Omega primality	
$\omega(N) < \infty$.	If S_r is atomic and $r \in \mathbb{Q} \cap (0,1)$, then $\omega(S_r) = \infty$.

References

- A. Assi and P. A. García-Sánchez: *Numerical Semigroups and Applications*, RSME Springer Series, Springer, New York, 2016.
- D. F. Anderson, S. T. Chapman, N. Kaplan, and D. Torkornoo: *An algorithm to compute ω-primality in a numerical monoid*, Semigroup Forum **82** (2011) 96–108.
- S. T. Chapman, M. Corrales, A. Miller, C. Miller, and D. Patel: *The catenary degrees of elements in numerical monoids generated by arithmetic sequences*, Comm. Algebra **45** (2017) 5443–5452.
- S. T. Chapman, J. Daigle, R. Hoyer, and N. Kaplan: *Delta sets of numerical monoids using nonminimal sets of generators*, Comm. Algebra **38** (2010) 2622–2634.
- P. A. García-Sánchez and J. C. Rosales: *Numerical Semigroups*, Developments in Mathematics Vol. 20, Springer-Verlag, New York, 2009.
- A. Geroldinger, W. A. Schmid, and Q. Zhong: (2017) *Systems of Sets of Lengths: Transfer Krull Monoids Versus Weakly Krull Monoids*. In: M. Fontana, S. Frisch, S. Glaz, F. Tartarone, P. Zanardo (eds) Rings, Polynomials, and Modules. Springer, Cham.

- F. Gotti: *Increasing positive monoids of ordered fields are FF-monoids*, J. Algebra **518** (2019) 40–56.
- F. Gotti: *On the atomic structure of Puiseux monoids*, J. Algebra Appl. **16** (2017) 1750126.
- F. Gotti: *On the system of sets of lengths and the elasticity of submonoids of a finite-rank free commutative monoid*, J. Algebra Appl. (to appear). [arXiv:1806.11273]
- F. Gotti: *Puiseux monoids and transfer homomorphisms*, J. Algebra **516** (2018) 95–114.
- F. Gotti: *Systems of sets of lenths of Puiseux monoids*, J. Pure Appl. Algebra **223** (2019) 1856–1868.
- F. Gotti and M. Gotti: *Atomicity and boundedness of monotone Puiseux monoids*, Semigroup Forum **96** (2018) 536–552.
- F. Gotti and C. O'Neil: *The elasticity of Puiseux monoids*, J. Commut. Algebra (to appear). [arXiv:1703.04207]
- M. Gotti: *On the local k-elasticities of Puiseux monoids*, Internat. J. Algebra Comput. **29** (2019) 147–158.