A characterization of non-Noetherian BFDS and FFDs

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March 25, 2019

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We say D is a BFD if D is atomic and if for all $b \in D^* \setminus U(D)$ there exists a $\pi(b) \in \mathbb{N}$, such that whenever $b = a_1 a_2 \cdots a_k$ is a factorization of b into a product of irreducibles (atoms) then $k \leq \pi(b)$.

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D is said to satisfy the ascending chain condition on principal ideals (ACCP), if every chain of strictly increasing principal ideals terminates.

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For $b \in D$, we will denote the set of maximal ideals that contain b by $\max(b)$.

Definition

Let D be an integral domain and let $b \in D^*$. We say $Z(b)$ is disconnected if there exists $\{a_i\}_{i=1}^{\infty} \subseteq Z(b)$ such that $max(a_i) \cap max(a_j) = \emptyset$ whenever $i \neq j$. We say $Z(b)$ is connected if it is not disconnected.

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Definition

We say an integral domain D is connected if for all $b \in D$, $Z(b)$ is connected. We will say D is disconnected if there exists $b \in D$ such that $Z(b)$ is disconnected.

Lemma

Let D be an integral domain and let $d \in D^*$ with $a, b \in Z(d)$. If $max(a) \cap max(b) = \emptyset$, then $ab \in Z(d)$.

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Proof.

We will use the fact that $D = \cap_{M \in \text{Max}(D)} D_M$. We first observe that both $\frac{d}{a}, \frac{d}{b} \in D_M$ for all $M \in \text{Max}(D)$. Now since $b \notin M$ for all $M \notin \text{max}(b)$ it is the case that $\frac{d}{ab} \in D_M$ for all $M \notin \text{max}(b)$. Now since $\frac{d}{b} \in D_M$ for all M and $a \notin M \in \text{max}(b)$ we have that $\frac{d}{ab} \in D_M$ for all $M \in \text{max}(b)$. Thus $\frac{d}{ab} \in D_M$ for all M. We conclude that $ab \in Z(d)$. □

Theorem

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Proof.

Suppose D is disconnected. Then there exists a $d \in D$ such that $Z(d)$ is disconnected. We find $\{a_i\}_{i=1}^{\infty} \subset Z(d)$ such that $\max(a_i) \cap \max(a_j) = \emptyset$ for all $i \neq j$. Now using the lemma we see that

$$(d) \subsetneq (\frac{d}{a_1}) \subsetneq (\frac{d}{a_1 a_2}) \subsetneq (\frac{d}{a_1 a_2 a_3}) \cdots$$

is an infinite strictly increasing chain of principal ideals. Hence D does not satisfy ACCP. □

Now since ACCP is a consequence of FFD and BFD, we see that FFDs and BFDs need to be connected. One might ask if connectedness is sufficient for any of these conditions. The answer is no, in fact a domain can be connected and not even be atomic.

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Example

The domain $D = \mathbb{Z}_{(2)} + x\mathbb{Q}[[x]]$ *is connected but is not atomic. To see this observe that* D *is quasi-local and* x *can never be factored as a finite product of atoms.*

Definition

Let D be an integral domain and let $b \in D^*$. We say $S = \{M_1, M_2, \cdots, M_k\} \subset max(b)$ is a finite covering of $Z(b)$ if for all $d \in Z(b)$ there exists an $i \in \{1, \cdots, k\}$ such that $d \in M_i$. We say D is finitely coverable if for all $b \in D^*$, $Z(b)$ has a finite covering.

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The previous example shows that an integral domain can be finitely coverable and yet fail to be atomic.

However, if D is almost Dedekind and finitely coverable then D is a BFD.

Let D be almost Dedekind and denote by ν_M the local valuation map from D_M into $\mathbb{N}_0$. Recall that if $b \in M$, then $\nu_M(b) > 0$ and $\nu_M(\frac{b}{d}) = \nu_M(b) - \nu_M(d)$.

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Theorem

Let D be an almost Dedekind domain. If D is finitely coverable, then D is a BFD.

Proof.

Let $b \in D^*$. Now find $S = \{M_1, M_2, \cdots, M_k\}$ that covers $Z(b)$. Now, since every divisor d of b is contained in some M_i, the value of $\frac{b}{d}$ is decreased by at least one in M_i. Thus $\pi(b) = \sum_{i=1}^{k} \nu_{M_i}(b)$ is a bound on the length of factorizations of b. □

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Now clearly if two elements b and c are in only finitely many maximal ideals, their product bc is in only finitely many maximal ideals. Further if $b \in \mathcal{F}$ and c divides b, we must have $b = cl$ for some l. It is clear from the equation that c can only be in finitely many maximal ideals. Thus $\mathcal{F}$ is a multiplicatively closed saturated set. Thus, in a one-dimensional integral domain, $\mathcal{F}$ must be the set compliment of the union of maximal ideals.

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So

$$\mathcal{F} = (\cup_{M \in M^{\infty}} M)^{c},$$

for some $M^{\infty} \subset \mathrm{Max}(D)$. Thus we see

$$\mathcal{F}^{c} = \cup_{M \in M^{\infty}} M.$$

That is if $b \in M$ for some $M \in M^{\infty}$ then $|\max(b)| = \infty$.

We partition the divisors of $b \in D$ along the same lines. More precisely let $Z^{\infty}(b) = \{d \in Z(b) : |\max(d)| = \infty\}$ and $Z^{F}(b) = \{d \in Z(b) : |\max(d)| < \infty\}$.

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Theorem

Let D be a connected domain. Then $Z^{F}(b)$ is finitely covered for all $b \in D^{*}$.

Proof.

Let $H = \{a_1, \cdots a_l\} \subset Z^{F}(b)$ be a set that is maximal with respect to $\max(a_1), \max(a_2), \cdots, \max(a_l)$ being mutually disjoint. We know this set must be finite, else D would be disconnected. Now set $S = \cup_{i=1}^{l} \max(a_i)$ and note that S is finite since each of the $\max(a_i)$ are finite. Further if $d|b$ we must have $d \in M$ for some $M \in S$ else H would not be maximal with respect to the $\max(a_i)$'s being mutually disjoint. □

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Theorem

Let D be an almost Dedekind domain with $M^{\infty} = \{M_1, M_2, \cdots, M_l\}$. The following are equivalent.

i) *D is connected*
ii) *D satisfies ACCP*
iii) *D is a BFD.*

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Let D be an almost Dedekind domain with $M^{\infty} = \{M_1, M_2, \cdots, M_l\}$. The following are equivalent.

i) *D is connected*

ii) *D satisfies ACCP*

iii) *D is a BFD.*

Proof.

Suppose D is connected. Then for all $b \in D$, $Z^F(b)$ can be finitely covered by some set S. Now $S \cup M^{\infty}$ is a finite covering of $Z(b)$. Thus D is a BFD. It is well known that BFD implies ACCP in any integral domain. We have already established that ACCP implies connected. □

Definition

Let D be an integral domain and $b \in D^*$. We say $Z(b)$ behaves finitely if $|Z_M(b)| < \infty$ for all $M \in max(b)$. We say an integral domain is finitely behaved if for all $b \in D^*$, $Z(b)$ behaves finitely.

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Definition

Let D be an integral domain and let $b \in D^*$. We say $Z(b)$ is l-bounded at $M \in max(b)$ if there exists $l_M \in \mathbb{N}$ such that given any $d_1, d_2, \cdots, d_{M_l} \in Z_M(b)$ the product $d_1 d_2 \cdots d_{l_M}$ does not divide b. Moreover we say $Z(b)$ is l_∞-bounded if there exists $l_\infty \in \mathbb{N}_0$ such that given any $d_1, d_2, \cdots, d_{l_\infty} \in Z^\infty(b)$ the product $d_1 d_2 \cdots d_{l_\infty}$ does not divide b.

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Definition

We say an integral domain D is l-bounded, if for all $b \in D^*$, $Z(b)$ is both l-bounded and l_∞-bounded.

Theorem

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Proof.

Suppose D is an FFD. It should be clear that D is finitely behaved. Let $b \in D^*$. Now b has only finitely many divisors, say $d_1, d_2, \cdots, d_k$. Choosing $M_1 \in \max(d_1), M_2 \in \max(d_2), \cdots, M_k \in \max(d_k)$, we see that $S = \{M_1, M_2, \cdots, M_k\}$ is a finite cover of $Z(b)$
Now suppose $Z(b)$ has a finite cover and is finitely behaved. Let $S = \{M_1, M_2, \cdots, M_k\}$ be a finite cover of $Z(b)$. Now $|Z(b)| \leq \sum_{i=1}^{k} |Z_{M_i}(b)|$ showing $Z(b)$ is finite. We conclude that D is an FFD. □

Theorem

Let D be an integral domain. D is a BFD if and only if D is connected and I-bounded.

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Proof.

It should be clear that if D is not connected or not l-bounded, then D is not a BFD. Suppose D is connected and l-bounded, and let $b \in D^*$. Since D is connected we have from Theorem 8 that $Z^F(b)$ is finitely covered, say by $\{M_1, M_2, \cdots, M_k\}$. Now the length of the factorization of b is less than or equal to $\pi(b) = l_\infty + \sum_{i=1}^{k} l_{M_i}$. Thus D is a BFD. □

An almost Dedekind domain D is said to be a sequence domain if $\text{Max}(D) = \{M_1, M_2, \cdots\} \cup M^*$ such that each M_i is principal and M^* is a dull maximal ideal. Now given $b \in D^*$, $\nu_{M_i}(b)$ and $\nu_{M^*}(b)$ are bounds showing that both $Z(b)$ is finitely behaved and I-connected. All sequence domains fail to be atomic. This shows that finitely behaved and I-bounded are not enough to force finite factorization or bounded factorization.

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