Infinite Product (수정4)

All rings are commutative with identity.

For a UFD D, D[[x]] need not be UFD. Samuel's counterexample.

Conjecture:

1. D[[x]] is a UFD if and only if D[[x]] is a GCD domain.

2. For a valuation domain V,
V[[x]] is a UFD if and only if V[[x]] is a GCD domain.

Theorem. Let V be a rank-one valuation domain. If V[[x]] is a GCD domain, then the infinite product of x-a $_i$ exists for all a$_i$ such that the sum of v(a$_i$) is finite.

Definition. Let a, and b be elements of a ring R. We say that a is the infinite product of a1,...an,... $a_1,\ldots,a_n,\ldots$ if

(1) a1...an $a_1 \cdots a_n$ divides a for all n.

(2) If a1...an $a_1 \cdots a_n$ divides b for all n, then a divides b.

Definition. A very weak UFR (unique factorization ring) is a ring, whose every element is either the finite product or the infinite product of prime elements.
A very weak UFD is a very weak UFR, which is an integral domain.

Definition. A ring R is a super weak UFR if it is a vw UFR and if for prime elements p1,…,pn,… $p_1,\ldots,p_n,\ldots$ of R, p1…pn $p_1\cdots p_n$ divides an element a of D for each n, there exists the infinite product of p1,…,pn,… $p_1,\ldots,p_n,\ldots$
A super weak UFD is a sw UFR, which is an integral domain.

Example. The ring of entire functions is a super weak UFD.

Remark. Every bounded descending sequence in a super weak UFR has a limit. In fact, every bounded descending sequence has a limit. It is either zero (that is, every element smaller than all the elements of the sequence is zero) or some nonzero element.

Theorem. Let a be the infinite product of prime elements a$_n$'s. Then a is the infinite product of any permutations of a$_n$'s.

Proof. Let σ be a permutation on the natural numbers. Then $a_{\sigma(1)}\cdots a_{\sigma(n)}$ is a part of $a_1\cdots a_m$, where m is bigger than $\sigma(1),\ldots,\sigma(n)$. So $a_{\sigma(1)}\cdots a_{\sigma(n)}$ divides a. Suppose that $a_{\sigma(1)}\cdots a_{\sigma(n)}$ divides b for all n. Let k be a natural number. Then

$a_1\cdots a_k$ = $a_{\sigma(\eta(1))}\cdots a_{\sigma(\eta(k))}$

, where η is the inverse function of σ, is a part of

$a_{\sigma(1)} \cdots a_{\sigma(l)}$

, where l > all $\eta(1), \ldots, \eta(k)$. So $a_1 \cdots a_k$ divides b for all k. So a divides b. So a is the infinite product of $a_{\sigma(1)}, \ldots, a_{\sigma(n)}, \ldots$

Theorem. The above representation in a domain is unique up to units.

Proof. Let a be the infinite product of prime elemets $p_1, \ldots, p_n, \ldots$ We show that p_n's are the only prime divisors of a. Let p be a prime dividing a and a=px. If p is not one of pn's p_n's, then each p1...pn $p_1 \cdots p_n$ divides x. So a divides x. Hence p is a unit, a contrdiction. Therefore p is one of the p_n's.

Example. ΠD_n , where each D_n is a UFD, is a UFR.

Example of a sw UFD, which is not a UFD. The ring of entire functions is such an example.

Theorem. A UFD is a sw UFD.

Theorem(?). Let D be a domain and S be the set of elements that are not infinite products of prime elements. Then S is a m.s. and D$_S$ D_S is a vw UFD.

Remark. accp need not hold for a vw UFR. The ring of entire functions.

A vw UFD might not be a GCD domain.

Theorem. A sw UFD is a GCD domain.

Corollary. A sw UFD is an integrally closed domain.

Theorem. The polynomial ring over a sw UFD is also a sw UFD.

Definition. A polynomial f is v-primitive if $(C_f)_v$ = 1.

Theorem. In a GCD domain, an irreducible v-primitive element is a prime element.

Remark. The power series ring over a sw UFD need not be a sw UFD. Note that the power series ring over a UFD is a UFD if and only if it is a GCD domain.

Questions.

1. D[[x]] is a UFD if and only if it is a sw UFD?
2. D[[x]] is a UFD if and only if it is a GCD domain?

Theorem. A valuation domain is a sw UFD if and only if it is a UFD.

Lemma. In a sw UFD, an irreducible element is a prime element.

Lemma. Let D be a sw UFD and S be an arbitrary countable subset of D. Then the lcm of S exists.

Theorem. A sw UFD is a pseudo proncipal domain, that is, all divisorial ideals are principal.

Corollary. If D is a sw UFD, then $(C_{fg})v = (C_f C_g)v$ for all f, g in D[[x]]. So D is completely integrally closed.

Theorem. **Let V be a rank-one valuation domain. If V[[x]] is a GCD domain, then the infinite product of x-a$_i$ exists for all a$_i$ such that the sum of v(a$_i$) is finite.**

Proof. Let a_i be such that **the sum of v(a_i) is finite. By choosing suitable increasing sequence of prime numbers p_i and q_i, we can define the infinite product f and g of (x^{p_i}-a^{p_i})** and (x^{q_i}-$a_i^{q_i}$). Let h be the gcd of f and g. Then each x-a_i divides h. So sum of v(a_i) $\leq$ v(h_0). Let s be a prime factor of f other than x-a_i. Then s cannot divide g/(x-a_i). (Note that any b in V[[x]] with unit content is a product of finitely many prime elements.) For otherwise f and g has a common prime factor, say t and in the extension domain V[[x]]/(t) of V, f=g=0. However x^{p_i}=a^{p_i} and x^{q_i}=a^{q_i} and hence x=a in V[[x]]/(t), which implies x-a $\in$ (t). So x-a=t, a contradiction. Thus s cannot divide g/(x-a_i). So s should divide g/h. So $v(h_0)$ = sum $v(a_i)$ hence h is the infinite product of $x-a_i$.

Lemma. Let V be a valuation domain and I be an ideal of V[[x]]. If I:x=I and f is an element of I such that v(f(0)) is the smallest among I, then I=(f).

Lemma. Let V be a rank-one valuation domain and f be a primitive prime element. Then V[[x]]/(f) is a 1-dimensional domain.

Lemma. Let V be a rank-one valuation domain and f be a nonunit primitive prime element. Then there is a rank-one valuation extension domain W of V, where f has a zero.

Proof. Every nonunit primitive power series f is a product of prime elements of V[[x]]. Let p(z) be a prime factor of f(z). Then V[[z]]/(p) is a 1-dimensional quasi-local domain, which is an extension ring of V. Let W be a complete rank-one valuation domain centered on the maximal ideal of V. Then z+(p) is a zero of f(x) in W[[x]]. QED.

Theorem. Let V be a rank-one valuation domain with value group R the real numbers. Then there is a rank-one (complete) valuation domain W centered on V such that every nonconstant nonunit power series has a zero in W and is the infinite product of infinitely many x-z.

Proof. Let f be a nonconstant nonunit power series. Choose a nonunit m so that its value is small enough to make f(mx) a nonconstant nonunit power series. Then f=ag, where a is in V and g is a nonunit nonconstant primitive power series. Choose a rank-one valuation extension domain of V, where g has a zero. Then f has a zero in W. Let S be the collection of rank-one valuation extensions of V. Partially order S by W=W' or W < W' iff W=W' or W' is an extension of W, W' is centered on W, there is a nonunit nonconstant power series f in W[[x]], which do not have a zero in W but in W'. We claim that S has a maximal element. Let Wn be a chain in S. Then the union is an upper bound for the chain Wn. Let W be a maximal element. Then every nonunit nonconstant power series of W has a zero in W. For otherwise W has a rank-one valuation extension W' centered on W, where f has a zero, and hence W < W', a contradiction. QED.

Definition. Let V be a rank-one valuation domain with value group R the real numbers. Then there is a rank-one (complete) valuation domain W centered on V such that every nonconstant nonunit power series has a zero in W and is the infinite product of infinitely many x-z. W is called a **power closure** of V.