Families of Numerical Semigroups

Kunz Coordinates and Semigroup Trees

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AMS 2019:
Factorization and Arithmetic Properties of Integral Domains and Monoids

March 22, 2019

Definition

A numerical semigroup S is an additive submonoid of $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, where $\mathbb{N}_0 \setminus S$ is finite. That is, $a, b \in S$ implies $a + b \in S$.

A numerical semigroup S has a unique minimal generating set $\{n_1, \ldots, n_t\}$. Elements of S are linear combinations of $n_1, \ldots, n_t$ with nonnegative integer coefficients:

$$S = \langle n_1, \ldots, n_t \rangle = \{a_1 n_1 + \cdots + a_t n_t \mid a_1, \ldots, a_t \in \mathbb{N}_0\}.$$

Definition

The size of the minimal generating set of S is the embedding dimension of S, denoted $e(S)$.

Example

$$\begin{aligned}
\mathbb{N}_0 = \langle 1 \rangle &= \{0, 1, 2, \ldots\}, \\
\langle 2, 3 \rangle &= \{0, 2, 3, 4, \ldots\}, \\
\langle 2, 5 \rangle &= \{0, 2, 4, 5, 6, \ldots\}, \\
\langle 4, 5, 6, 7 \rangle &= \{0, 4, 5, 6, 7, 8, \ldots\}, \\
\langle 3, 5, 7 \rangle &= \{0, 3, 5, 6, 7, 8, \ldots\}.
\end{aligned}$$

Definition

1. *The smallest nonzero element of S is the multiplicity of S, denoted $m(S)$.*
2. *The elements of the complement $\mathbb{N}_0 \setminus S$ are the gaps of S. The largest gap is the Frobenius number of S, denoted $F(S)$.*
3. *The number of gaps is called the genus of S, denoted $g(S)$.*

Example

S	$m(S)$	$\mathbb{N}_0 \setminus S$	$F(S)$	$g(S)$
$\langle 2,3\rangle$	2	$\{1\}$	1	1
$\langle 2,5\rangle$	2	$\{1,3\}$	3	2
$\langle 3,4,5\rangle$	3	$\{1,2\}$	2	2
$\langle 2,7\rangle$	2	$\{1,3,5\}$	5	3
$\langle 3,4\rangle$	3	$\{1,2,5\}$	5	3
$\langle 4,5,6,7\rangle$	4	$\{1,2,3\}$	3	3
$\langle 3,5,7\rangle$	3	$\{1,2,4\}$	4	3
$\langle 3,7,8\rangle$	3	$\{1,2,4,5\}$	5	4
$\langle 3,8,10\rangle$	3	$\{1,2,4,5,7\}$	7	5
$\langle 3,7,11\rangle$	3	$\{1,2,4,5,8\}$	8	5

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$\langle 3,5,7\rangle$	3	$\{1,2,4\}$	4	3
$\langle 3,7,8\rangle$	3	$\{1,2,4,5\}$	5	4
$\langle 3,8,10\rangle$	3	$\{1,2,4,5,7\}$	7	5
$\langle 3,7,11\rangle$	3	$\{1,2,4,5,8\}$	8	5
$\langle 2,2g+1\rangle$	2	$\{1,3,5,\ldots,2g-1\}$	$2g-1$	g
$\langle g+1,g+2,\ldots,2g+1\rangle$	$g+1$	$\{1,2,\ldots,g\}$	g	g
$\langle a,b\rangle$	a		$ab-a-b$	$\frac{(a-1)(b-1)}{2}$

Some Major Problems about Numerical Semigroups

Question (Frobenius Problem)

Let $S = \langle n_1, \ldots, n_t \rangle$.
Can we give a 'nice' formula for $F(S)$ in terms of $n_1, \ldots, n_t$?

For example, when $S = \langle a, b \rangle$,

$$F(S) = ab - a - b, \quad \text{and} \quad g(S) = \frac{(a-1)(b-1)}{2}.$$

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For example, when $S = \langle a, b \rangle$,

$$F(S) = ab - a - b, \quad \text{and} \quad g(S) = \frac{(a-1)(b-1)}{2}.$$

Let $N(g)$ be the number of numerical semigroups S with $g(S) = g$.

Question (Counting Semigroups by Genus)

How fast does $N(g)$ *grow?*
Is it an increasing function of g*?*

g	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$N(g)$	1	1	2	4	7	12	23	39	67	118	204	343	592	1001	1693	2857

The Wilf Conjecture

Conjecture (Wilf, 1978)

For any numerical semigroup S,

$$\frac{g(S)}{F(S)+1} \leq 1 - \frac{1}{e(S)}.$$

Idea: If $g(S)$ is not too much smaller than $F(S)+1$, then S must have many generators.

Conjecture

The number of small elements of S, those less than $F(S)$, is denoted $n(S)$. We have

$$e(S)n(S) \geq F(S)+1.$$

Idea: The number of small elements and the number of minimal generators cannot simultaneously be small.

The Weight of a Numerical Semigroup

Definition

Let S be a numerical semigroup with gap set $\mathbb{N}_0 \setminus S = \{l_1, \ldots, l_g\}$.

The weight of S is

$$w(S) = \sum_{i=1}^{g} (l_i - i).$$

Example

1. Let $S = \langle 3, 7, 8 \rangle$.
   $\mathbb{N}_0 \setminus S = \{1, 2, 4, 5\}$, so
   $$w(S) = (1 + 2 + 4 + 5) - (1 + 2 + 3 + 4) = 2.$$
2. Let $S = \langle 3, 8, 10 \rangle$.
   $\mathbb{N}_0 \setminus S = \{1, 2, 4, 5, 7\}$, so
   $$w(S) = (1 + 2 + 4 + 5 + 7) - (1 + 2 + 3 + 4 + 5) = 4.$$

Effective Weight

Definition

Let S be a numerical semigroup with gap set $\mathbb{N}_0 \setminus S = \{l_1, \ldots, l_g\}$.

The *effective weight* *of S is*

$$ewt(S) = \sum_{l \in \mathbb{N}_0 \setminus S} \#\{minimal\ generators\ a < l\}.$$

$ewt(S) = \#\{pairs\ (a, b) \colon 0 < a < b,\ \ a\ is\ a\ generator\ and\ b\ is\ a\ gap\}.$

Example

1. Let $S = \langle 3, 7, 8 \rangle$, so $\mathbb{N}_0 \setminus S = \{1, 2, 4, 5\}$.

$$ewt(S) = 0 + 0 + 1 + 1 = 2.$$

2. Let $S = \langle 3, 8, 10 \rangle$, so $\mathbb{N}_0 \setminus S = \{1, 2, 4, 5, 7\}$.

$$ewt(S) = 0 + 0 + 1 + 1 + 1 = 3.$$

Examples and Pflueger's Conjecture

S	$m(S)$	$\mathbb{N}_0 \setminus S$	$F(S)$	$g(S)$	$\frac{g(S)}{F(S)+1}$	$1-\frac{1}{e(S)}$	$ewt(S)$
$\langle 2,3\rangle$	2	$\{1\}$	1	1	1/2	1/2	0
$\langle 2,5\rangle$	2	$\{1,3\}$	3	2	1/2	1/2	1
$\langle 3,4,5\rangle$	3	$\{1,2\}$	2	2	2/3	2/3	0
$\langle 3,5,7\rangle$	3	$\{1,2,4\}$	4	3	3/5	2/3	1
$\langle 3,7,8\rangle$	3	$\{1,2,4,5\}$	5	4	2/3	2/3	2
$\langle 3,8,10\rangle$	3	$\{1,2,4,5,7\}$	7	5	5/8	2/3	3
$\langle 3,7,11\rangle$	3	$\{1,2,4,5,8\}$	8	5	5/9	2/3	4
$\langle 2,2g+1\rangle$	2	$\{1,3,\ldots,2g-1\}$	$2g-1$	g	1/2	1/2	$g-1$
$\langle g+1,\ldots,2g+1\rangle$	$g+1$	$\{1,2,\ldots,g\}$	g	g	$g/(g+1)$	$g/(g+1)$	0
$\langle a,b\rangle$	a		$ab-a-b$	$\frac{(a-1)(b-1)}{2}$	1/2	1/2	

$$ewt(\langle a,b\rangle) = (a-1)(b-1) - a - b + \left\lfloor \frac{b}{a} \right\rfloor + 2.$$

Conjecture (Pflueger,2018)

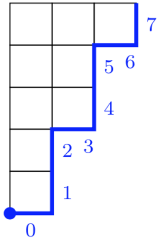
Let S be a semigroup with $g(S) = g$. Then

$$ewt(S) \leq \left\lfloor \frac{(g+1)^2}{8} \right\rfloor.$$

Main Idea: The Enumeration of S

We create a partition $\lambda(S)$ called the enumeration of S by walking along the outer profile of the partition.

Start at 0: Step Right if $i \in S$ and Step Up if $i \notin S$.

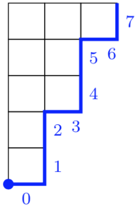


$\lambda(S)$ for $S = \langle 3, 8, 10\rangle = \{0, 3, 6, 8, 9, 10, \ldots\}$.

$\mathbb{N}_0 \setminus S = \{1, 2, 4, 5, 7\}$.
The size of $\lambda(S)$ is $w(S) + g(S) = 4 + 5 = 9$.

The Enumeration of S: Examples

$$\langle 3, 8, 10\rangle = \{0, 3, 6, 8, 9, 10, \ldots\}.$$

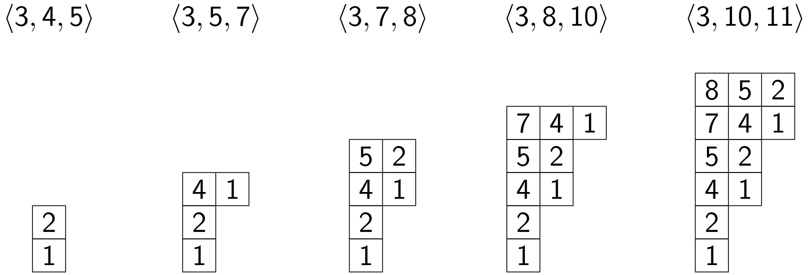


Definition

For each box in a partition there is a hook length, *the number of boxes strictly below it, plus the number of boxes to the right of it, plus* 1.

7	4	1
5	2	
4	1	
2		
1		

$\lambda(S)$ and Wilf's Conjecture

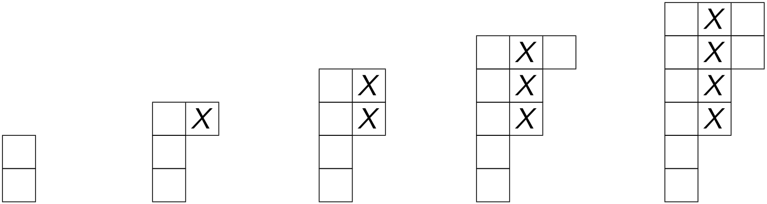


Length of first column: $g(S)$.
Length of first row: $n(S)$.
Largest hook length: $F(S)$.
Length of first row plus length of first column: $F(S) + 1$.

Wilf's Conjecture:
If the first column of $\lambda(S)$ is much larger than its first row, $e(S)$ is large.

$\lambda(S)$ and Pflueger's Conjecture

$\langle 3,4,5\rangle$ $\quad$ $\langle 3,5,7\rangle$ $\quad$ $\langle 3,7,8\rangle$ $\quad$ $\langle 3,8,10\rangle$ $\quad$ $\langle 3,10,11\rangle$



$$ewt(\langle 3,5,7\rangle) = 1, \qquad ewt(\langle 3,7,8\rangle) = 2,$$
$$ewt(\langle 3,8,10\rangle) = 3, \qquad ewt(\langle 3,10,11\rangle) = 4.$$

Pflueger's Conjecture:
$\lambda(S)$ cannot have too many boxes above its minimal generators relative to the length of its first column.

The Semigroup Tree

The Semigroup Tree is a rooted tree with root $\mathbb{N}_0$.
Nodes at level g correspond to semigroups of genus g.

For a numerical semigroup S of genus g,
$S' = S \cup \{F(S)\}$ is a numerical semigroup of genus $g-1$.
Note that $F(S) > F(S')$.
Adjoining $F(S')$ to S' gives a semigroup of genus $g-2$, and so on.

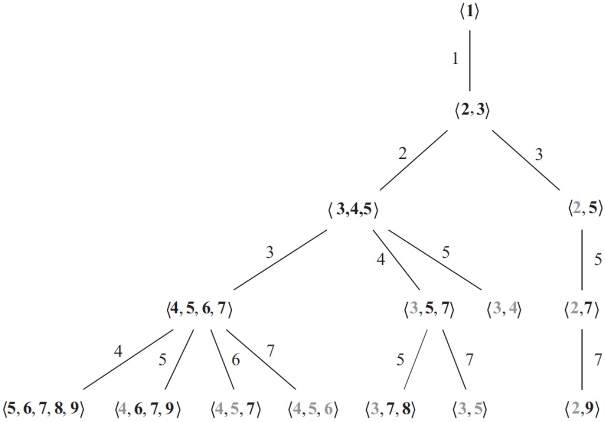
Starting from S we get a path of $g+1$ semigroups, one of each genus $g' \leq g$, ending at $\mathbb{N}_0$.

Definition

The effective generators of S are the elements of its minimal generating set that are larger than $F(S)$.

The children of S are the numerical semigroups of genus $g+1$ that come from removing an effective generator from S.

The Semigroup Tree



A generator of a semigroup is in gray if it is not greater than $F(S)$.
An edge between S and its child S' is labeled by x if $S' = S \setminus \{x\}$.

Figure from [Fromentin-Hivert, 2016]

$\lambda(S)$ and the Semigroup Tree

If S' is a child of S in the semigroup tree, how are $\lambda(S')$ and $\lambda(S)$ related?

$\langle 3,4,5\rangle$

2
1

$\langle 3,5,7\rangle$

4	1
2	
1	

$\langle 3,7,8\rangle$

5	2
4	1
2	
1	

$\langle 3,8,10\rangle$

7	4	1
5	2	
4	1	
2		
1		

$\langle 3,10,11\rangle$

8	5	2
7	4	1
5	2	
4	1	
2		
1		

If $F(S) > m(S)$ (which is true in most cases), add a new top row.

We have $e(S') = e(S)$ or $e(S) - 1$.

$ewt(S')$ increases by the number of minimal generators of $e(S)$ less than the one we removed.

Kunz Coordinate Vectors / Apéry Tuples

A numerical semigroup S containing m contains an element in each nonzero residue class modulo m.

Let $k_i m + i$ be the minimum element of S congruent to i modulo m.

Definition

The Apéry set of S (with respect to m) is

$$Ap(S, m) = \{0, k_1 m + 1, k_2 m + 2, \ldots, k_{m-1} m + (m-1)\}.$$

The Apéry tuple or Kunz coordinate vector of S is

$$KC(S, m) = (k_1, \ldots, k_{m-1}) \in \mathbb{N}_0^{m-1}.$$

S	$\langle 3,4,5 \rangle$	$\langle 3,5,7 \rangle$	$\langle 3,7,8 \rangle$	$\langle 3,8,10 \rangle$	$\langle 3,10,11 \rangle$
$KC(S,3)$	$(1,1)$	$(2,1)$	$(2,2)$	$(3,2)$	$(3,3)$

Question

If Wilf's/Pflueger's Conjecture holds for the semigroup corresponding to $KC(S, m)$, does it also hold for 'nearby' semigroups?

The geometry of Kunz Coordinates

Not every vector in $\mathbb{N}_0^{m-1}$ is the Kunz coordinate vector of a numerical semigroup containing m.

Example:

$(1,3)$ is not the Kunz coordinate vector of a semigroup containing 3.

Theorem (Branco, García-García, García-Sánchez, Rosales, 2002)

There is a bijection between:

1. *Numerical semigroups with multiplicity m and genus g,*
2. *Integer solutions $(k_1, \ldots, k_{m-1})$ to the inequalities:*

$$\begin{aligned} x_i &\geq 1 && \text{for all} \quad i \in \{1, \ldots, m-1\}, \\ x_i + x_j &\geq x_{i+j} && \text{for all} \quad 1 \leq i \leq j \leq m-1,\ i+j \leq m-1, \\ x_i + x_j + 1 &\geq x_{i+j-m} && \text{for all} \quad 1 \leq i \leq j \leq m-1,\ i+j > m, \end{aligned}$$

with

$$\sum_{i=1}^{m-1} x_i = g.$$