Leamer Monoids and the Huneke-Wiegand Conjecture

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Christopher O'Neill
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The Huneke-Wiegand Conjecture

In *Tensor Products of Modules and the Rigidity of Tor*:

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Huneke-Wiegand Conjecture (1994)

Let R be a one-dimensional Gorenstein domain. Let $M \neq 0$ be a finitely-generated R-module, which is not projective. Then the torsion submodule of $M \otimes_R \operatorname{Hom}_R(M, R)$ is non-trivial.

Numerical Semigroup Rings

Let $\mathbb{K}$ be a field and Γ be a numerical semigroup. A **numerical semigroup ring** $\mathbb{K}[\Gamma]$ is the subring of $\mathbb{K}[t]$ given by

$$\bigoplus_{s \in \Gamma} k_s t^s,$$

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Goal: Use semigroup structure of Γ to prove the Huneke-Wiegand conjecture for certain ideals of $\mathbb{K}[\Gamma]$.

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$$(n_1, m_1) + (n_2, m_2) =$$

$$\{n_1, n_1+s, \ldots, n_1+m_1 s\} + \{n_2, n_2+s, \ldots, n_2+m_2 s\} =$$

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$\Rightarrow$ If $(n_1, m_1), (n_2, m_2) \subset \Gamma$, then $(n_1+n_2, m_1+m_2) \subset \Gamma$.

Leamer Monoids

Definition

Given a numerical monoid Γ and a gap element $s \in \Gamma$, the set

$$S_\Gamma^s = \{(n,m) \,:\, \{n, n+s, \ldots, n+ms\} \subset \Gamma\}$$

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Irreducible elements: An arithmetic sequence $(n, m) \in S_\Gamma^s$ is *irreducible* if it cannot be written as the sum of two other non-trivial arithmetic sequences.

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- Reducible: $(28, 3) = \{28, 31, 34, 37\} = \{7, 10\} + \{21, 24, 27\} = (7, 1) + (21, 2)$
- Irreducible: $(57, 2) = \{57, 60, 63\}$

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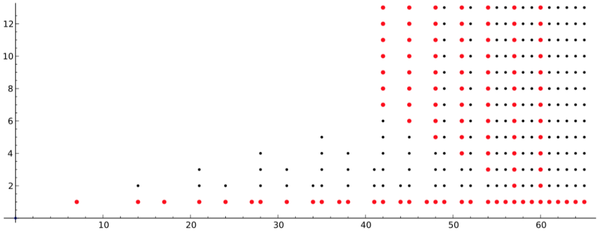

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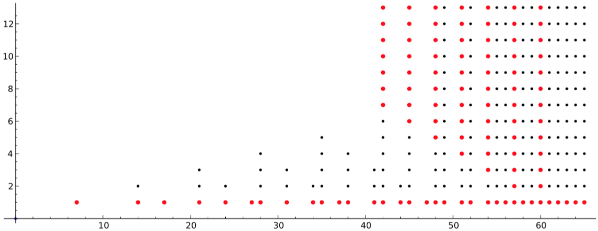

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Examples:

- $(28, 3) = (7, 1) + (21, 2)$ is reducible
- $(57, 2)$ is irreducible

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Theorem [García-Sanchez, Leamer]

Let Γ be a symmetric numerical semigroup and $\mathbb{K}$ a field. The numerical semigroup ring $\mathbb{K}[\Gamma]$ satisfies the Huneke-Wiegand conjecture for 2-generated monomial ideals if and only if, for each gap element $s \in \mathbb{N} \setminus \Gamma$, there exists an irreducible arithmetic sequence $(x, 2) \in S_\Gamma^s$.

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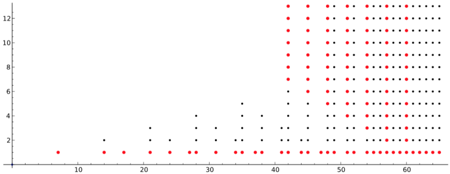
- There exists an irreducible $(x, 2) \in S_\Gamma^s$ iff $I = \langle t^a, t^{a+s} \rangle$ satisfies the HW conjecture for any $a \in \Gamma$.
- For $R = \mathbb{K}[\Gamma]$, the length of the torsion submodule of $I \otimes_R \mathrm{Hom}_R(I, R)$ is the number of height 2 irreducible elements in S_Γ^s.

Finding Irreducible Elements of Height 2

Goal: For *every* gap element $s \notin \Gamma$, find a height 2 irreducible.



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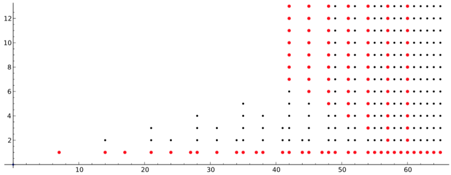

Problem: This graph only corresponds to one gap element s.

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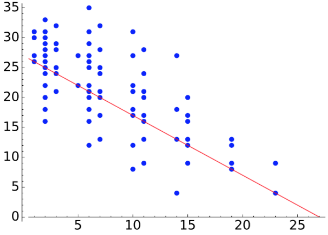

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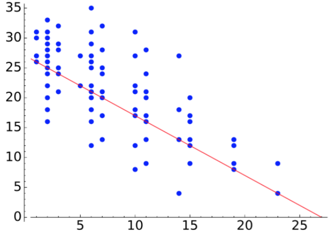

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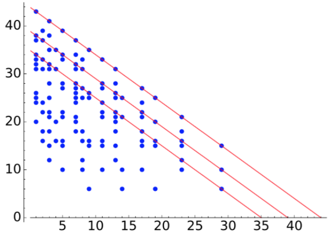
Example: $\langle 4, 9 \rangle$ will have a height 2 irreducible at $n = 27 - s$.

Embedding Dimension 3

Consider $\Gamma = \langle 6, 10, 15 \rangle$.



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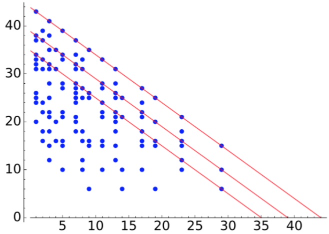

We need to use multiple lines!

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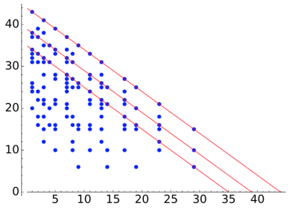

Theorem

Given a symmetric numerical semigroup $\Gamma = \langle n_1, n_2, n_3 \rangle$ with embedding dimension 3, S_Γ^s has a height 2 irreducible at $\mathcal{F}(\Gamma) - s + n_j$ for some minimal generators n_j.

$\mathcal{F}(\Gamma) =$ Frobenius number of Γ.

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Let n, d, h, k be positive integers. A numerical monoid generated by a **generalized arithmetic sequence** is of the form

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For example, when $n = 6, d = 1, h = 2, k = 4$, we obtain

$$\langle 6, 13, 14, 15, 16 \rangle.$$

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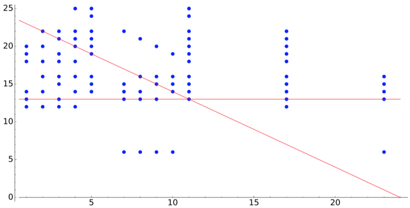
Consider $\Gamma = \langle 6, 13, 14, 15, 16 \rangle$.



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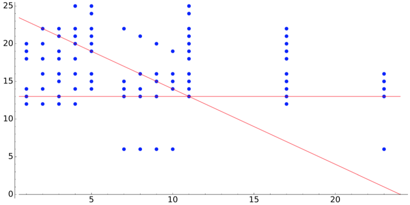

Here, we need one non-constant linear function and one constant function.

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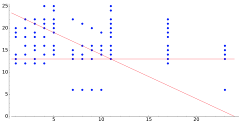

Theorem

For a numerical semigroup $\Gamma = \langle n, nh + d, nh + 2d, \ldots, nh + kd \rangle$ generated by a generalized arithmetic sequence, there exists a height 2 irreducible at $\mathcal{F}(\Gamma) - s + d$ or $nh + d$.

Thank you!

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