A characterization of Krull monoids for which sets of lengths are arithmetical progressions

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Sets of lengths

A monoid H (commutative, cancellative), for example the multiplicative monoid of a domain, is called

1. *atomic* if each non-zero element a is the product (of finitely many) irreducible elements.
2. *factorial* if there is an essentially unique factorization into irreducibles (i.e., up to ordering and associates).

If the structure is not factorial, one still wants to "understand" the arithmetic.

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Sets of lengths, II

For example, study *sets of lengths*.

If

$$a = a_1 \dots a_n$$

with irred. a_i, then n is called a length of a.

$$\mathsf{L}(a) = \{n \colon n \text{ is a length }\}.$$

For a invertible set $\mathsf{L}(a) = \{0\}$.
The *system of sets of lengths* is

$$\mathcal{L}(H) = \{\mathsf{L}(a) \colon a \in H\}.$$

In general, sets of lengths can be infinite. Yet, for Krull monoids, Dedekind domains, numerical monoids, ... they are *finite*.
So

$$\mathcal{L}(H) \subset \mathbb{P}_{\text{fin}}(\mathbb{N}_0).$$

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General properties of systems of sets of lengths (of BF)

We have

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What else?
Let $L, L' \in \mathcal{L}(H)$.

- If $0 \in L$, then $L = \{0\}$.
- If $1 \in L$, then $L = \{1\}$.
- Let $S = L + L' = \{l + l' : l \in L, l' \in L'\}$. There exists some $L'' \in \mathcal{L}(H)$ such that $S \subset L''$.

We have $\mathsf{L}(a) + \mathsf{L}(b) \subset \mathsf{L}(ab)$.

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Direct consequences:

- $\{\{0\}\} \subset \mathcal{L}(H)$ and equality holds if and only if H is a group.
- If H is not a group, then $|\mathcal{L}(H)|$ infinite.
- If $\mathcal{L}(H)$ contains some L with $|L| \geq 2$, then $\mathcal{L}(H)$ contains arbitrarily large sets.

Moreover

$$\mathcal{L}(H) \subset \{\{0\}, \{1\}\} \cup \mathbb{P}_{\text{fin}}(\mathbb{N}_{\geq 2}).$$

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Structure Theorem of Lengths (Geroldinger, 1988)

Theorem

Let H be a Krull monoid where only finitely many classes contain prime divisors. Then there exists a finite set Δ^* and some M such that for each $a \in H$ the set $\mathsf{L}(a)$ is an AAMP with difference $d \in \Delta^*$ and bound M.

Almost arithmetical multiprogression

We say, L is an AAMP if

$$L = y + (L' \cup L^* \cup L'') \subset y + \mathcal{D} + d\mathbb{Z}$$

with

- $\{0, d\} \subset \mathcal{D} \subset [0, d]$ (period)
- $L^* = [0, l''] \cap (\mathcal{D} + d\mathbb{Z})$ (central part)
- $L' \subset [-M, -1]$ and $L'' \subset [l' + 1, l' + M]$ (initial and end part)

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Example of an AAMP

10, 12, 14, 16, 18,
20, 22, 24, 25, 26, 28,
30, 32, 34, 35, 36, 38,
40, 42, 44, 45, 46, 48,
50, 55, 56

Difference: 10
Period: $\{0, 2, 4, 5, 6, 8, 10\}$
Bound: 10

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Are almost arithmetical multiprogression necessary?

A sort of converse to STSL (S. 2009)

Theorem

Let $M \in \mathbb{N}_0$ and $\emptyset \neq \Delta^* \subset \mathbb{N}$ finite. Exists a finite abelian group G s.t.:
for every AAMP L with difference $d \in \Delta^*$ and bound M there is some $y_{G,L}$ such that

$$y + L \in \mathcal{L}(G) \text{ for all } y \geq y_{G,L}.$$

Explicit version

$M \in \mathbb{N}$ and $\emptyset \neq \Delta^* \subset \mathbb{N}$, $D = \max \Delta^*$. Let G be a finite abelian group. $\mathcal{L}(G)$ contains (up to shift) each AAMP with difference $d \in \Delta^*$ and bound M if

- G has a subgroup of the form

$$\Big(\bigoplus_{j=1}^{r} \langle e_j \rangle\Big) \oplus \langle f \rangle \oplus \bigoplus_{d \in \Delta^*} \Big(\oplus_{i=0}^{\lceil (M+d-1)/d \rceil} \langle e_i^d \rangle\Big),$$

  where $r \geq 12(M^2 + D)$, $\operatorname{ord} e_j \geq 5$, $\operatorname{ord} f \geq 24(M^2 + D)$ and $\operatorname{ord} e_i^d = d(\lceil (M + d - 1)/d \rceil + i) + 2$, or
- for some prime $p \geq 5$ the p-rank of G is at least $21(M^2 + D)$.

These groups are 'large.' Can we do better for restrained classes of groups?

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Remarks regarding "large"

Theorem (Infinite class group (Kainrath, 1999))

Let H be a Krull monoid with class group G such that each class contains a prime divisor.
If G is infnite then

$$\mathcal{L}(H) = \{\{0\}, \{1\}\} \cup \mathbb{P}_{fin}(\mathbb{N}_{\geq 2}).$$

In other words $\mathcal{L}(H)$ is as large as possible. Or:
"Every" set is is a set of lengths.

1. More explicit and refined investigations by Baginski, Rodriguez, Schaeffer, She.
2. 'Asymptotic' version by Geroldinger, S., Zhong.

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Block monoid $\mathcal{B}(G_0)$

Block monoids are an important class of auxiliary monoids. They allow to investigate sets of lengths, for (transfer) Krull monoids, which includes numerous examples of interest (see Geroldinger's talk).
Let $(G, +, 0)$ be an abelian group.
Let $G_0 \subset G$. A *sequence* S over G_0 is an element of $\mathcal{F}(G_0)$ the free abelian monoid with basis G_0.
Thus a sequences is a (formal, commutative) product

$$S = \prod_{i=1}^{l} g_i = \prod_{g \in G_0} g^{\mathsf{v}_g(S)}.$$

The sequence S is called a *zero-sum sequence* if its *sum*

$$\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G_0} \mathsf{v}_g(S) g \in G$$

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Let H be a Krull monoid where only finitely many classes contain prime divisors. Then there exists a finite set Δ^* and some M such that for each $a \in H$ the set $\mathsf{L}(a)$ is an AAMP with difference $d \in \Delta^*$ and bound M.

Problem: Can we do better for certain class groups? (Let us assume that every class contains a prime divisor.)
Answer: Obviously. Assume the class group has order 1 ...
Or also, if the class group has order at most 2, then every set of lengths is a singleton. (Carlitz, 1960).

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Theorem (STSL)

Let H be a Krull monoid where only finitely many classes contain prime divisors. Then there exists a finite set Δ^* and some M such that for each $a \in H$ the set $\mathsf{L}(a)$ is an AAMP with difference $d \in \Delta^*$ and bound M.

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Let G be some group. What type of sets of lengths can we construct (easily)?

Arithmetic progressions.

Suppose $\mathrm{ord}(g) = n$ then

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Almost arithmetic progressions. Suppose $\mathrm{ord}(g_1) = 4$, $\mathrm{ord}(g_2) = 5$, then

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Small class group (Geroldinger 1990)

Let H be a Krull monoid with class group G such that each class contains a prime divisor.

If $G = C_3$ then

$$\mathcal{L}(H) = \{y + 2k + [0, k] \colon y, k \in \mathbb{N}_0\}$$

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When are AAMPs not necessary?

(Geroldinger–S.)

Theorem

The following statements are equivalent:

(a) *There is a constant $M \in \mathbb{N}$ such that all sets of lengths in $\mathcal{L}(G)$ are AAPs with bound M.*

(b) *G is a subgroup of C_4^3 or a subgroup of C_3^3.*

We say, L is an AAP if

$$L = y + (L' \cup L^* \cup L'') \subset y + d\mathbb{Z}$$

with

- $L^* = [0, l'] \cap (\mathcal{D} + d\mathbb{Z})$ (central part)
- $L' \subset [-M, -1]$ and $L'' \subset [l' + 1, l' + M]$ (initial and end part)

In other words an AAMP with period $\{0, d\}$.

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The following statements are equivalent:

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Intrinsic structural properties of $\mathcal{L}(G)$

Recall: Let $S = L + L' = \{l + l' \colon l \in L,\ l' \in L'\}$. There exists some $L'' \in \mathcal{L}(H)$ such that $S \subset L''$.

We say that $\mathcal{L}(H)$ is additively closed if for all $L, L' \in \mathcal{L}(H)$ there exists some $L'' \in \mathcal{L}(H)$ such that

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Theorem (Geroldinger–S. 2014)

Let H be a Krull monoid with class group G and suppose that each class contains a prime divisor. Then the system of sets of lengths $\mathcal{L}(H)$ is additively closed under set addition if and only if G has one of the following forms:

(a) *G is cyclic of order $|G| \le 4$.*

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(c) *G is an elementary 3-group of rank $r \le 2$.*

(d) *G is infinite.*

Intrinsic structural properties of $\mathcal{L}(G)$

Recall: Let $S = L + L' = \{l + l' : l \in L, \ l' \in L'\}$. There exists some $L'' \in \mathcal{L}(H)$ such that $S \subset L''$.
We say that $\mathcal{L}(H)$ is additively closed if for all $L, L' \in \mathcal{L}(H)$ there exists some $L'' \in \mathcal{L}(H)$ such that

$$L + L' = L''.$$

Theorem (Geroldinger–S. 2014)

Let H be a Krull monoid with class group G and suppose that each class contains a prime divisor. Then the system of sets of lengths $\mathcal{L}(H)$ is additively closed under set addition if and only if G has one of the following forms:

(a) *G is cyclic of order $|G| \le 4$.*

(b) *G is an elementary 2-group of rank $r \le 3$.*

(c) *G is an elementary 3-group of rank $r \le 2$.*

(d) *G is infinite.*

When are AAMPs not necessary?, IV

Theorem

The following statements are equivalent

(a) *All sets of lengths in $\mathcal{L}(G)$ are arithmetical progressions.*

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A characterization of Krull monoids for which sets of lengths are arithmetical progressions

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