Class groups of cluster algebras

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FWF Der Wissenschaftsfonds.

Cluster algebras

Cluster algebras were introduced by Fomin and Zelevinsky in 2002.

More than 600 preprints on the arXiv.

Connections to many different area of mathematics: Total positivity, combinatorics, Teichmüller theory, representation theory, knot theory, Lie algebras, ...

Defined via combinatorial data: Quivers and mutations.

Quiver:

- finite directed graph
- (for us:) no loops or 2-cycles
- parallel arrows allowed.

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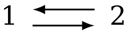

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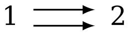

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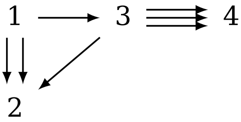
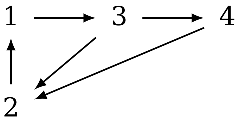


acyclic

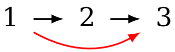


with cycle(s)

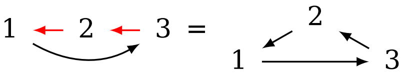
Quiver mutations I

Mutation of a quiver Q **at vertex** i.

1. For arrows $j \to i \to k$, add arrows $j \to k$.



2. Flip all arrows incident with i.



3. Remove 2-cycles.

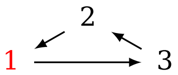
✓

Parallel mutation of **seed**: In $\{x_1, \ldots, x_n\}$ replace x_i by x_i':

$$x_i x_i' = \prod_{j \to i} x_j + \prod_{i \to j} x_j.$$

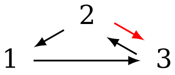
$$\left\{x_1, x_2, x_3\right\} \rightsquigarrow \left\{x_1, \frac{x_1 + x_3}{x_2}, x_3\right\}$$

Quiver mutations II



$\{x_1, \frac{x_1+x_3}{x_2}, x_3\}$

1. For arrows $j \to i \to k$, add arrows $j \to k$.



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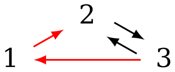
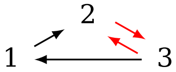


3. Remove 2-cycles.



$\rightsquigarrow \quad 1 \leftarrow 2 \leftarrow 3$

$x_1 x_1' = x_2' + x_3$, so new seed $\left\{\frac{x_1+(1+x_2)x_3}{x_1 x_2}, \frac{x_1+x_3}{x_2}, x_3\right\}$.

Cluster algebras

Let Q be a quiver on vertices $\{1, \ldots, n\}$ and $\{x_1, \ldots, x_n\}$ an initial seed.

- Mutation yields a (possibly infinite) collection of seeds.
- Each element of a seed is a **cluster variable**.

Definition

The **cluster algebra** $A = A(Q)$ is the subalgebra of $\mathbb{Z}(x_1, \ldots, x_n)$ generated by all cluster variables.

$$\mathbb{Z}[x_1, \ldots, x_n] \subset A \subset \mathbb{Z}(x_1, \ldots, x_n).$$

Cluster algebras

Example (A_3)

$$A_3\colon \quad 1 \longrightarrow 2 \longrightarrow 3$$

$$\begin{aligned}A(A_3) = \mathbb{Z}\Big[x_1, x_2, x_3, \frac{1+x_2}{x_1}, \frac{x_1+x_3}{x_2}, \frac{1+x_2}{x_3},\\ \frac{x_1+(1+x_2)x_3}{x_1x_2}, \frac{(1+x_2)x_1+x_3}{x_2x_3},\\ \frac{(1+x_2)(x_1+x_3)}{x_1x_2x_3}\Big]\end{aligned}$$

Laurent phenomenon

Theorem (Fomin, Zelevinsky 2002)

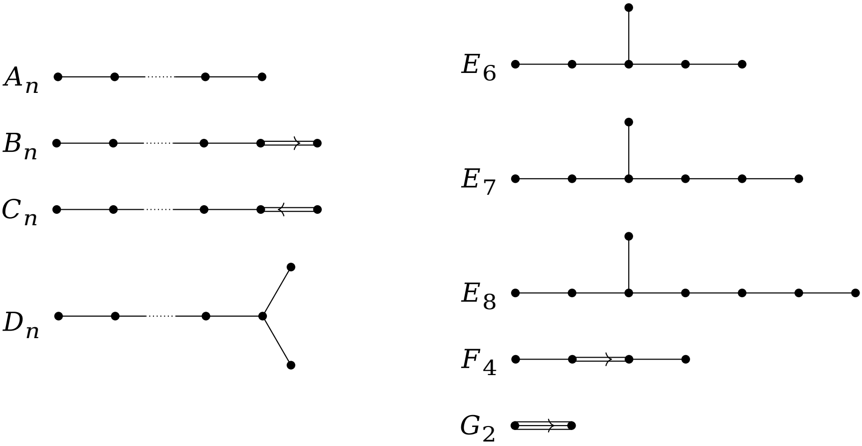
Denominators of cluster variables are monomials, hence

$$A \subseteq \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

Finite type classification

Theorem (Fomin, Zelevinsky 2003)

Cluster algebras of **finite type** (=having finitely many cluster variables) are classified by Dynkin diagrams.



Goal

Understand factorizations of elements into **atoms** (**irreducible elements**) in cluster algebras.

1. When is $A(Q)$ **factorial** (a **UFD**)?
2. What happens if it is not?

Factorizations: earlier results

Theorem (Geiß, Leclerc, Schröer, 2012)

1. Cluster variables are (pairwise non-associated) atoms.
2. If A is factorial, all exchange polynomials $f_i \in \mathbb{Z}[x_1, \ldots, x_n]$ with $x_i x_i' = f_i$ are irreducible and pairwise distinct.

Example

- If $f_i = g_1 \cdots g_k$, then $x_i x_i' = g_1 \cdots g_k$.
- If $f_i = f_j$ for $i \neq j$, then $x_i x_i' = x_j x_j'$.

Theorem (Lampe, 2012, 2014)

Classification of factoriality for simply-laced Dynkin types (A_n, D_n, E_n).

Acyclic cluster algebras

x_i'... obtained from initial seed $\{x_1, \ldots, x_n\}$ by mutation at i:

$$x_i x_i' = f_i \quad \text{with} \quad f_i = \prod_{j \to i} x_j + \prod_{i \to j} x_j \quad \text{exchange polynomials.}$$

Theorem (Berenstein, Fomin, Zelevinsky 2006; Muller 2014)

Let Q be **acyclic**. Then

$$A = \mathbb{Z}[x_1, x_1', \ldots, x_n, x_n'] \cong \mathbb{Z}[X_1, X_1', \ldots, X_n, X_n']/(X_i X_i' - f_i).$$

A is **finitely generated**, **noetherian**, **integrally closed**.

Corollary

(Locally) acyclic cluster algebras are **Krull domains**.

Krull domains

Theorem

Let A be a Krull domain with **divisor class group** $G = \mathcal{C}(A)$ and

$$G_0 = \{[\mathfrak{p}] : \mathfrak{p} \text{ divisorial [=height-1] prime}\} \subseteq G.$$

Then there exists a **transfer homomorphism**

$$\varphi \colon (A \setminus \{0\}, \cdot) \to \mathcal{B}(G_0),$$

with $\mathcal{B}(G_0)$ the **monoid of zero-sum sequences** over G_0.

Corollary

1. A is **factorial** (= a **UFD**) if and only if G is trivial.
2. Factorization theory of A determined by G and G_0.

First Main Result

Theorem (Garcia Elsener, Lampe, S., 2017)

Let $A = A(Q)$ be a Krull domain (e.g., Q acyclic), and $\{x_1, \dots, x_n\}$ a seed. Then

- $G = \mathcal{C}(A) \cong \mathbb{Z}^r$ for some $r \geq 0$, and every class contains infinitely many prime divisors ($G_0 = G$).
- A is factorial if and only if $r = 0$.
- $r = t - n$ with t the number of height-1 primes containing one of $x_1, \dots, x_n$.

Consequences

Corollary

For Q acyclic[1], the necessary conditions of Geiß–Leclerc–Schöer are sufficient for $A(Q)$ to be factorial.

Corollary

Acyclic cluster algebras with (invertible) **principal coefficients** are factorial.

Corollary

If $A = A(Q)$ is a Krull domain but **not** factorial, then Kainrath's Theorem applies: for every $L \subseteq \mathbb{Z}_{\geq 2}$ there exists $a \in A$ with $\mathsf{L}(a) = L$.

[1] without isolated vertices

Goal

Get explicit description of the rank r of $\mathcal{C}(A) \cong \mathbb{Z}^r$, directly in terms of Q.

Restrict to Q acyclic.

Exchange matrix

For a quiver Q, define

- **Signed adjacency matrix:** skew-symmetric $n \times n$-matrix $B = B(Q)$, with

$$b_{ij} = \#\{\text{arrows } i \to j\} - \#\{\text{arrows } j \to i\}.$$

- a vector $d \in \mathbb{Z}^n$ with d_i the gcd of the i-th column of B.

Partners

Definition

Vertices i, $j \in [1, n]$ are **partners** if the following equivalent conditions hold.

1. Exchange polynomials f_i, f_j have a common factor.
2. there exist **odd** c_i, $c_j \in \mathbb{Z}$: $c_j b_{*i} = c_i b_{*j}$.
3. $\mathrm{v}_2(d_i) = \mathrm{v}_2(d_j)$ and $b_{*i}/d_i = \pm b_{*j}/d_j$.

Partnership is an equivalence relation on $[1, n]$: **Partner sets**.

Example

Example

$$1 \longrightarrow 2 \longrightarrow 3$$

$$B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$d = \begin{pmatrix} 1, & 1, & 1 \end{pmatrix}$$

Partner sets: $\{1, 3\}$, $\{2\}$.

Main result for acyclic quivers

For a partner set $V \subseteq [1, n]$ and $d \geq 1$, let

$$\mathrm{c}(V, d) = \#\{i \in V \mid d \text{ divides } d_i\}.$$

(Recall: d_i is gcd of the i-th column of adjacency matrix B)

Theorem (Garcia Elsener, Lampe, S. 2017)

Let Q be **acyclic** and $A = A(Q)$. Then $\mathcal{C}(A) \cong \mathbb{Z}^r$ with

$$r = \sum_{V \text{ a partner set}} r_V,$$

where

$$r_V = \sum_{\substack{d \geq 1 \\ d \text{ odd}}} (2^{\mathrm{c}(V,d)} - 1) - \#V.$$

Corollary: finite type

Corollary

If Q is **acyclic** and **without parallel arrows**, then $A(Q)$ is factorial if and only if there are no partners $i \neq j$.

Corollary

For the cluster algebras of Dynkin types:

- Type A_n is factorial if $n \neq 3$, and $\mathcal{C}(A_3) \cong \mathbb{Z}$.
- Type B_n is factorial if $n \neq 3$, and $\mathcal{C}(B_3) \cong \mathbb{Z}$.
- Type C_n is factorial.
- Type D_n has $\mathcal{C}(D_n) \cong \mathbb{Z}$ for $n > 4$, and $\mathcal{C}(D_4) \cong \mathbb{Z}^4$.
- Types E_6, E_7, and E_8 are factorial.
- Type F_4 is factorial.
- Type G_2 has $\mathcal{C}(G_2) \cong \mathbb{Z}$.

Summary

- For cluster algebras that are Krull domains, the **class group is always of form $\mathbb{Z}^r$.**
- For **acyclic** cluster algebras, r can be expressed directly in terms of the quiver and is **trivial to compute**.

Similar results hold

- over fields of characteristic 0 as ground ring, and
- for skew-symmetrizable cluster algebras with (invertible) frozen variables.

Open questions

- How to determine r in the locally acyclic case?
- When is $A(Q)$ a Krull domain? [completely] integrally closed?