Patterns of ideals of numerical semigroups

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Numerical semigroups

Denote $\mathbb{Z}_+$ the set of non-negative integers.

A **numerical semigroup** is a subset $S \subset \mathbb{Z}_+$, such that

- S is closed under addition,
- $0 \in S$ and
- the complement $(\mathbb{Z}_+) \setminus S$ is finite.

Numerical semigroups

The **multiplicity** of a numerical semigroup is its smallest non-zero element.

The **conductor** of a numerical semigroup is the smallest element such that all subsequent natural numbers belong to the numerical semigroup (the Frobenius number $+1$).

The **gaps** of a numerical semigroup are the elements in the complement of the numerical semigroup.

A **relative ideal** I of a numerical semigroup S is a set $I \subseteq \mathbb{Z}$ satisfying

- $I + S \subseteq I$
- $I + d \subseteq S$ for some $d \in S$.

An **ideal** is a relative ideal contained in S (so $d = 0$).

An ideal is **proper** if it is distinct from S.

Examples

- The maximal ideal of a numerical semigroup S is $M(S) = S \setminus \{0\}$. It is maximal among the proper ideals of S.
- Let $S = \langle 3, 7 \rangle = \{0, 3, 6, 7, 9, 10, 12, \dots\}$:
  - $H = S \setminus \{0, 6\}$ is NOT an ideal,
  - $I = S \setminus \{0, 7\} = \{3, 6, 9, 10, 12, \dots\}$ is an ideal,
  - $I - 7 = \{-4, -1, 2, 3, 5, \dots\}$ is a relative ideal.

The **dual** of a relative ideal H is the relative ideal

$$H^* = (S - H) := \{x \in \mathbb{Z} : x + H \subseteq S\}.$$

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Definition. [Bras-Amorós and García-Sánchez, 2006]

A **homogeneous pattern** admitted by a numerical semigroup S is a homogeneous linear multivariate polynomial $p = \sum_{i=1}^{n} a_i X_i$ such that $p(s_1, \dots, s_n) \in S$ for all non-increasing sequences $s_1, \cdots, s_n \in S$.

Examples.

- Arf numerical semigroups are characterized by admitting the homogeneous linear "Arf pattern" $X_1 + X_2 - X_3$.
- Homogeneous linear patterns of the form $X_1 + \cdots + X_k - X_{k+1}$ generalise the Arf property and are called **subtraction patterns** [Bras-Amorós and García-Sánchez, 2006].

But with this definition of pattern all **non-homogeneous patterns** must have constant term in S.

$$p(0,\dots,0)=\sum_{i=1}^{n} a_i\cdot 0+a_0=a_0\in S.$$

To overcome this problem, when the non-homogeneous patterns were introduced it was with $M(S)$ as domain [Bras-Amorós, García-Sánchez, and Vico-Oton,2013]. But now we have two different definitions of patterns.
Let us generalise and unify!

Definition.
A **pattern** admitted by an ideal I of a numerical semigroup S is a multivariate polynomial function which returns an element in S when evaluated on any non-increasing sequence of elements from I.

We say that the ideal I **admits** the pattern.

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If $I = S$, then we say that the numerical semigroup S admits the pattern.

What happened with the previous definitions of patterns?

- Homogeneous patterns evaluted on S have become patterns admitted by S.
- Non-homogeneous patterns evaluated on $M(S)$ have become patterns admitted by $M(S)$.

Note that a pattern admitted by an ideal I of a numerical semigroup S is also admitted by any ideal $J \subseteq I$.

We identify the pattern with its polynomial.

We say that the pattern is **linear** and **homogeneous**, when the pattern polynomial is linear and homogeneous.

- The **length** of a pattern: the number of indeterminates.
- The **degree** of a pattern: the degree of the pattern polynomial.

One pattern p **induces** another pattern q if any ideal of a numerical semigroup that admits p also admits q.

Two patterns are **equivalent** if they induce each other.

Example. Consider the Arf pattern $p_{Arf}(X_1, X_2, X_3) = X_1 + X_2 - X_3$. It induces $q(X_1, X_2) = p_{Arf}(X_1, X_1, X_2) = 2X_1 - X_2$. It was proved by Campillo, Farrán and Munuera that q and p_{Arf} are equivalent.

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Theorem

Let $p(X_1, \ldots, X_n) = a_1X_1 + \cdots + a_nX_n$ be a homogeneous linear pattern admitted by $\mathbb{Z}_+$ and let I be an ideal of a numerical semigroup S. Then $p(I)$ is an ideal of some numerical semigroup if and only if $\gcd(a_1, \ldots, a_n) = 1$.

In particular, if $p(X_1, \ldots, X_n) = a_1X_1 + \cdots + a_nX_n$ is a homogeneous linear pattern admitted by $\mathbb{Z}_+$ and S a numerical semigroup, then $p(S)$ is a numerical semigroup if and only if $\gcd(a_1, \ldots, a_n) = 1$.

Lemma

If I is an ideal of some numerical semigroup S, then there is a $c \in I$ such that $z \in I$ for all $z \in \mathbb{Z}$ with $z \geq c$.

We call this c the maximum of the small elements of I.

Theorem

Let

- *I be an ideal of a numerical semigroup,*
- *$p(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_i X_i$ a homogeneous linear pattern with $\gcd(a_1, \ldots, a_n) = d (\geq 1)$ and let*
- *$b_1, \ldots, b_n$ (non-unique) integers such that $a_1 b_1 + \cdots + a_n b_n = d$.*

Then $J = p(I)/d$ is an ideal of a numerical semigroup.

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Then $J = p(I)/d$ is an ideal of a numerical semigroup.

- *Let $c(J)$ be the maximum of the small elements of J*
- *and let $\alpha = \sum_{i=1}^{n} a_i / d$.*

Then $c(J) < p(s_1, \ldots, s_n)/d$ whenever $s_n \geq c(I) - \min(0, (\alpha - 1) b_n)$ and $s_i \geq s_j + \max(0, (\alpha - 1)(b_j - b_i))$ for $1 \leq i < n$.

Therefore, the set of non-increasing sequences of I which is needed for calculating explicitly $p(I)$ is finite.

A linear pattern $p(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_i X_i + a_0$ is called *strongly admissible* if the partial sums $\sum_{i=1}^{n'} a_i \geq 1$ for all $1 \leq n' \leq n$.

I have written an algorithm for calculating $p(I)$ when p is strongly admissible. This algorithm is available in the numerical semigroup package *NumericalSgps* of GAP.

The image of linear patterns are (essentially) ideals.

Do all ideals and all numerical semigroups appear in this way?

Proposition

Any numerical semigroup $S = \langle a_1, \ldots, a_e \rangle$ *is the image of* $\mathbb{Z}_+$ *under the homogeneous pattern* $p(X_1, \ldots, X_e) = a_1 X_1 + \sum_{i=2}^{e} (a_i - a_{i-1}) X_i$.

So it is possible to define a numerical semigroup in terms of a pattern!

If the numerical semigroup S' is the image of a numerical semigroup $S \supseteq S'$ under a pattern p admitted by S, then S' admits p.

Consider the chain of numerical semigroups $S \supseteq p(S) \supseteq p\,(p\,(S)) \supseteq \cdots$.

If this chain stabilizes, then it does so immediately and $S = p(S)$.

Otherwise, what can we say about how S and $p(S)$ relate?

The quotient of a numerical semigroup S by a positive integer d is the numerical semigroup $\frac{S}{d} = \{x \in \mathbb{Z}_+ : dx \in S\}$.

Lemma

Let S be a numerical semigroup and let $p(X_1, X_2) = a_1X_1 + a_2X_2$ be a linear homogeneous pattern in two variables (not necessarily admitted by S) such that $a_1 \in S$ and $\gcd(a_1, a_2) = 1$. Then $S = \frac{p(S)}{a_1+a_2}$.

Corollary

Any numerical semigroup S is the quotient from division by d of infinitely many numerical semigroups of the form $p(S)$ for some pattern p, for any $d \in \mathbb{Z}$, $d \geq 2$.

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Definition.
Let p be a linear pattern admitted by an ideal I of a numerical semigroup. If $p(I) \subseteq I$, then we say that p is an **endopattern** of I.

Definition.
A pattern admitted by an ideal I with codomain J is **surjective** if $p(I) = J$.

So a surjective endopattern is a pattern of I such that $p(I) = I$.

A linear pattern $p(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_i X_i + a_0$ is *premonic* if $\sum_{i=1}^{n'} a_i = 1$ for some $n' \leq n$.

Lemma.
Any linear surjective endopattern of a proper ideal I of a semigroup S is necessarily of the form

$$p(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_i X_i + a_0$$

satisfying

$$a_0 = -(\sum_{i=1}^{n} a_i - 1)\mu(I)$$

where $\mu(I)$ is the smallest element of I. Also, if p is a premonic endopattern of I, such that

$$a_0 = -(\sum_{i=1}^{n} a_i - 1)\mu(I),$$

then p is surjective.

Now consider chains of ideals using pattern that are NOT admitted by the ideal:

Closures of **numerical semigroups** with respect to **homogeneous** patterns were introduced by [Bras-Amorós and García-Sanchez, 2006].

A pattern is admissible if it is admitted by some numerical semigroup.

Definition.
Given an ideal I of a numerical semigroup S and an admissible pattern p not necessarily admitted by I, define the **closure** of I with respect to p as the smallest ideal $\tilde{I}$ of some numerical semigroup $\tilde{S}$ that admits p and contains I.

Theorem

If

- I *is an ideal of a numerical semigroup and*
- $p(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_i X_i + a_0$ *is a premonic linear pattern satisfying* $a_0 = -(\sum_{i=1}^{n} a_i - 1)\mu$ *with* $\mu = \min(I)$.

then $I \subseteq p(I)$ *and the chain*

$$I_0 = I \subseteq I_1 = p(I_0) \subseteq I_2 = p(I_2) \subseteq \cdots$$

stabilizes. The ideal $I_k = p^k(I)$ *for* k *such that* $p^{k+1}(I) = p^k(I)$ *is the closure of* I *with respect to* p.

Example.

The closure of an Arf pattern is a numerical semigroup with the Arf property.

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An integer $x \notin S$ is **pseudo-Frobenius** if $s + x \in S$ for all $s \in S$.

So the polynomial $X_1 + a_0$ is a pattern admitted by S if and only if $a_0 \in S \cup PF(S)$, where $PF(S)$ is the set of pseudo-Frobenius.

Let

- $dM = \{m_1 + \cdots + m_d : m_i \in M\}$ and
- $(S - dM) := \{x \in \mathbb{Z} : x + dM \subseteq S\}$ (the dual of dM).

Definition.
For $d \geq 1$, define the set

$$PF^d(S) = (S - dM) \setminus (S - (d-1)M)$$

and call it **the set of elements at distance** d **from** S. Also, define $PF^0(S) = S$.

The elements at distance 0 from S are S.
The elements at distance 1 from S are $PF^1(S) = PF(S)$.
The elements at distance 2 from S are the elements x so that the pattern $X_1 + X_2 + x \in S$ for $X_1, X_2 \in M(S)$.
Etc.

Proposition

When S is of maximal embedding dimension, then $PF^2(S) = E(S) - 2m(S)$, where $E(S)$ is the set of minimal generators and $m(S)$ is the multiplicity.

The Lipman semigroup of S is $L(S) = \cup_{h \geq 1}(hM(S) - hM(S))$.

Theorem

The cardinality of $PF^d(S)$ converges to $m(S)$. The convergence follows the convergence of the Lipman semigroup of S.

Example. For $S = \langle 3, 5 \rangle = \{0, 3, 5, 6, 8, \dots\}$, with gaps $G = \{1, 2, 4, 7\}$:

- $PF^0(S) = S$
- $PF^1(S) = PF(S) = \{7\}$
- $PF^2(S) = \{2, 4\}$
- $PF^3(S) = \{-1, 1\}$
- $PF^4(S) = \{-4, -3, -2\}$
- $PF^5(S) = \{-7, -6, -5\}$
- $\dots$

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Lemma.
Let I be an ideal of a numerical semigroup S and suppose that p and q are two patterns admitted by I. Then

- $p + q$ and
- rp

are also patterns admitted by I for any polynomial r with coefficients in $\mathbb{Z}$ such that $r(I) \geq 0$ when evaluated on any non-increasing sequence of elements from I.

Denote by $\mathcal{P}_n^d(I)$ the set of patterns of length at most n and degree at most d, admitted by the ideal I of a numerical semigroup S.

Then $\mathcal{P}_n^1(I)$ is the set of linear patterns of length at most n admitted by I.

Proposition.
Let I be an ideal of a numerical semigroup S. Then

- $\mathcal{P}_n^d(I)$ is a **semigroup with zero, a monoid**.
- if $I \neq S$, then $\mathcal{E}_n^d(I)$ is a **semigroup without zero**.

In general we still have no nice characterization of $\mathcal{P}_n^d(I)$. We know something about the linear patterns of length 1.

Proposition.
For any numerical semigroup S we have

$$\mathcal{P}_1^1(S) \supseteq \{p(X_1) = a_1X_1 + a_0 \in \mathbb{Z}[X_1] : a_1 \geq 0, a_0 \in S \cup \bigcup_{i=1}^{a_1} PF^i(S)\}.$$

A numerical semigroup S is **ordinary** if $z \in S$ for all $z \in \mathbb{Z}$ such that $z \geq m(S)$.

Proposition.
If S is an ordinary numerical semigroup, then

$$\mathcal{P}_1^1(S) = \{p(X_1) = a_1X_1 + a_0 \in \mathbb{Z}[X_1] : a_1 \geq 0, a_0 \in S \cup \bigcup_{i=1}^{a_1} PF^i(S)\}.$$

Examples of applications of patterns:

- Numerical semigroups associated to combinatorial (r, k)-configurations admit:
  - $X_1 + X_2 - n$ for $n \in 0, \dots, \gcd(r, k)$, and
  - $X_1 + \cdots + X_{rk/\gcd(r,k)} + 1$.
- Weierstrass semigroups S of multiplicity $m(S)$ of a rational place of a function field over a finite field of cardinality q admit:
  - $qX_1 - qm(S)$ if the Geil-Matsumoto bound and the Lewittes bound coincide, and
  - $(q-1)X_1 - (q-1)m(S)$ if and only if the Beelen-Ruano bound equals $1 + (q-1)m$.

Thank you very much for listening!

Example: The numerical semigroup of parameters of combinatorial configurations

Incidence geometry

An **incidence geometry** (of rank 2) is a triple (P, L, I) where

- P is a set of 'points',
- L is a set of 'lines' ('blocks'),
- I is an incidence relation between the elements in P and L.

The **incidence graph** of the incidence structure (P, L, I) is the bipartite graph with vertex set $P \cup L$ and an edge between the vertices p and b if p is a point in b.

The incidence graph and the incidence structure contain the same information.

Combinatorial configurations

A **combinatorial** (v, b, r, k)-**configuration** is an incidence geometry with v points and b lines such that

- every point is on r lines,
- every line has k points,
- every pair of points is in at most one line, or equivalently,
- every pair of lines intersect in at most one point.

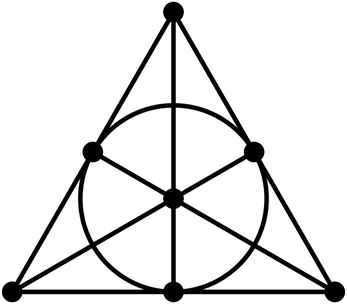
The four parameters (v, b, r, k) satisfy the relation $vr = bk$. So there is redundancy: three parameters are enough!

Reduced parameters: (d, r, k) with

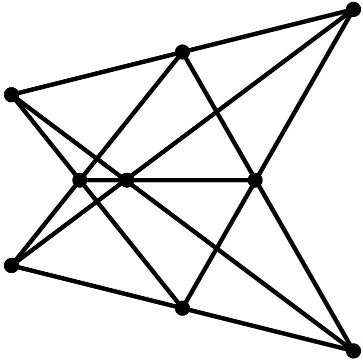
$$d := \frac{v \gcd(r,k)}{k} = \frac{b \gcd(r,k)}{r} = \frac{vr}{\operatorname{lcm}(r,k)} = \frac{bk}{\operatorname{lcm}(r,k)}.$$

Balanced configurations

We say that a combinatorial configuration is **balanced** if $r = k$. This implies that the number of points equals the number of lines and also, the associated integer, so $d = v = b$.



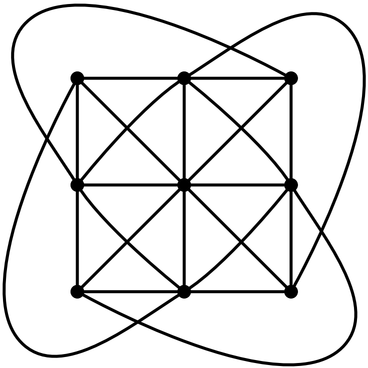
The Fano plane,
$(v, b, r, k) = (7, 7, 3, 3)$
$(d, r, k) = (7, 3, 3)$



The Pappus configuration,
$(v, b, r, k) = (9, 9, 3, 3)$
$(d, r, k) = (9, 3, 3)$

Non-balanced configurations

When $r \neq k$, then $v \neq b$ and $d = \frac{v\,\gcd(r,k)}{k}$.



The affine plane over $\mathbb{F}_3$

$$(v, b, r, k) = (9, 12, 4, 3)$$
$$(d, r, k) = (3, 4, 3)$$

The existence problem

Given a tuple (d, r, k), does there exist a (d, r, k)-configuration?

The following necessary conditions for existence of configurations are well-known.

Lemma

Suppose that there exists a (v, b, r, k)-configuration. Then

1. $v \geq r(k-1)+1$ *and* $b \geq k(r-1)+1$, *and*
2. $vr = bk$.

What about sufficient conditions?

- When $r = 3$, the necessary conditions are sufficent [Gropp (1994)].
- When r is larger, things get more complicated!

The existence problem

$r = k$	π	$v = b$							
3	7	7	$\rightarrow$						
4	13	13	$\rightarrow$						
5	21	21	~~22~~[1]	23	$\rightarrow$				
6	31	31	~~32~~[2]	~~33~~[3]	34	$\rightarrow$			
7	43	~~43~~[4]	~~44~~[1]	45	?46?	?47?	48	$\rightarrow$	
8	57	57	~~58~~[1]	?59?	?60?	?61?	?62?	63	$\rightarrow$
9	73	73	~~74~~[5]	?75?	?76?	?77?	78	?79?	80 $\rightarrow$

For unbalanced configurations in general less is known!

[1] [Bose and Connor (1952)]
[2] [Schellenberg (1975)]
[3] [Kaski and Östergård (2007)]
[4] [Bose (1938)]
[5] [Gropp (1992)])

Associating a numerical semigroup to combinatorial configurations

Parameter sets of combinatorial configurations

For which parameter sets do (v, b, r, k)-configurations exist?
We saw that a (v, b, r, k)-configuration has reduced parameter set (d, r, k) with

$$d = \frac{v \gcd(r, k)}{k} = \frac{b \gcd(r, k)}{r},$$

and we say that (d, r, k) is **configurable** if there is a configuration with these parameters.

The set of (r, k)-configurable tuples

Define $S_{(r,k)} = \{d \in \mathbb{Z}_+ : (d, r, k) \text{ is configurable}\}$.

Theorem (Bras-Amorós and S., 2012 (2009))

For every pair of integers $r, k \geq 2$, $S_{(r,k)}$ forms a numerical semigroup.

Sketch of proof.

Lemma

A set of positive integers generate a numerical semigroup if and only if they are coprime.

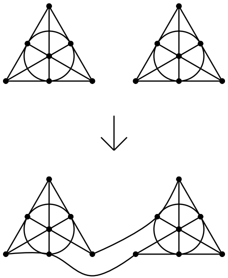
It is therefore enough to prove:

- $0 \in S_{(r,k)}$,
- $S_{(r,k)}$ is closed under addition,
- at least two elements of $S_{(r,k)}$ are coprime.

For the first fact, consider the empty configuration.

The two latter facts are proved by combining several configurations into larger configurations.

Addition



In terms of points and lines:

$$(\mathcal{P}_1, \mathcal{L}_1, \mathcal{I}_1) \oplus (\mathcal{P}_2, \mathcal{L}_2, \mathcal{I}_2) = (\mathcal{P}_1 \cup \mathcal{P}_2, \mathcal{L}_1 \cup \mathcal{L}_2, \mathcal{I})$$

In terms of reduced parameters (d, r, k):

$$d, d' \in S_{(r,k)} \Rightarrow d + d' \in S_{(r,k)}$$

.

Two coprime elements

- We want to construct two coprime elements in $S_{(r,k)}$.
- We get one element in $S_{(r,k)}$ (say d) associated to the combinatorial configuration obtained by taking parallel classes of a finite affine plane.
- We can construct a second combinatorial configuration with an associate integer that is coprime with d.
  We will see that we can (for example) do $2d - 1$.

Adding up we get a numerical semigroup X (with finite complement)

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Adding up we get a numerical semigroup X (with finite complement)

$$X = \{0,\ d,\ 2d-1,\ 2d,\ 3d-1, \ldots, 2(d-1)^2, \rightarrow\}.$$

This implies that $S_{(r,k)}$ has finite complement since $X \subseteq S$.

Patterns of numerical semigroups

Motivating example: Two constructions for balanced configurations

When $r = k$, then there are two constructions (see [Grünbaum]) implying

$$d_1, d_2 \in S_{(r,r)} \Rightarrow d_1 + d_2 - 1$$

and

$$d_1, d_2 \in S_{(r,r)} \Rightarrow d_1 + d_2 + 1.$$

Given an element $d \in S_{(r,r)}$ we get $2d - 1, 2d, 2d + 1 \in S_{(r,r)}$.

In particular this is enough for proving finite complement of $S_{(r,r)}$ in $\mathbb{Z}_+$.

Definition.

A **linear pattern** admitted by a numerical semigroup S is a linear multivariate polynomial $p = \sum_{i=1}^{n} a_i X_i + a_0$ such that $p(s_1, \ldots, s_n) \in S$ for all non-increasing sequences $s_1, \cdots, s_n \in S$.

Example.

A numerical semigroup has the **Arf property** if $s_1 + s_2 - s_3 \in S$ for every triple $s_1 \geq s_2 \geq s_3 \in S$.

Linear (homogeneous) patterns appeared as a generalisation of the Arf property.

A numerical semigroup is Arf iff it admits the pattern

$$p(X_1, X_2, X_3) = X_1 + X_2 - X_3.$$

Theorem (S. and Bras-Amorós, 2013)

Let $S_{(r,k)}$ be a numerical semigroup associated to the (r,k)-configurations. Then $S_{(r,k)}$ admits the pattern

$$X_1 + X_2 - n$$

for all $n \in [1, \ldots, \gcd(r,k)]$.

Construction.

- Take two (r,k)-configurations A and B with v_A and v_B points and b_A and b_B lines, respectively.
- Remove $a := nk/\gcd(r,k)$ points and $b := nr/\gcd(r,k)$ lines and match missing incidences.
- Obtain an (r,k)-configuration with $v = v_A + v_B - a$ points and $b = b_A + b_B - b$ lines. It has parameter $d = d_A + d_B - n$.

Theorem (S. and Bras-Amorós, 2013)

Let $S_{(r,k)}$ be a numerical semigroup associated to the (r,k)-configurations. Then $S_{(r,k)}$ admits the pattern

$$X_1 + X_2 - n$$

for all $n \in [1, \ldots, \gcd(r,k)]$.

Construction.

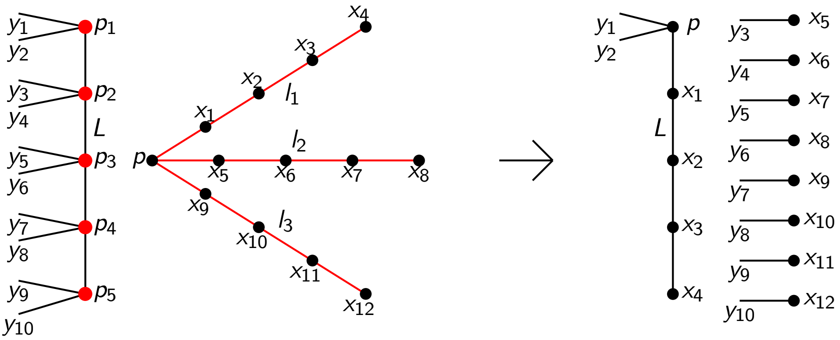
- Take two (r,k)-configurations A and B with v_A and v_B points and b_A and b_B lines, respectively.
- Remove $a := nk/\gcd(r,k)$ points and $b := nr/\gcd(r,k)$ lines and match missing incidences.
- Obtain an (r,k)-configuration with $v = v_A + v_B - a$ points and $b = b_A + b_B - b$ lines. It has parameter $d = d_A + d_B - n$.

How?

Example

An example of this construction for $(r, k) = (3, 5)$.

In this case $\gcd(r, k) = 1$, so the only possible choice of n is $n = 1$.



The red points and lines in the two combinatorial configurations on the left are removed and the resulting configuration is shown on the right.

Theorem (S. and Bras-Amorós, 2013)

The numerical semigroup $S_{(r,k)}$ admits the pattern

$$X_1 + \cdots + X_n + 1$$

with $n = rk/\gcd(r,k)$.

Proof.

- Take n combinatorial configurations $C_1, \ldots, C_n$ with reduced parameter sets $(d_1, r, k), \ldots, (d_n, r, k)$.
- On each configuration C_i, remove one point-line incidence (p_i, l_i).
- Instead let the n lines l_i all meet in sets of r in $k/\gcd(r,k)$ new points $p'_1, \ldots, p'_{k/\gcd(r,k)}$, and join the n points in sets of k over $r/\gcd(r,k)$ new lines $l'_1, \ldots, l'_{r/\gcd(r,k)}$.
- The configurations C_i have $v_i = d_i k/\gcd(r,k)$ points and $b_i = d_i r/\gcd(r,k)$ lines, the new configuration has parameters $(v_1 + \cdots + v_n + k/\gcd(r,k), b_1 + \cdots + b_n + r/\gcd(r,k), r, k)$, i.e. reduced parameters $(d_1 + \cdots + d_n + 1, r, k)$, so the numerical semigroup $S_{(r,k)}$ admits the pattern $X_1 + \cdots + X_n + 1$.

Why are we interested in linear patterns?

Theorem (S. and Bras-Amorós, 2013)

Let S be a numerical semigroup and

- *m the multiplicity of S,*
- *c the conductor of S,*
- *M the maximum integer such that S admits the pattern $x_1 + x_2 - a$ for $a \in \{1, \dots, M\}$.*
- *N the minimum integer such that S admits $x_1 + \dots + x_N + 1$.*

Then

$$c \leq \left\lceil \frac{m-1}{1/N + M} \right\rceil (m - M) + M.$$

For numerical semigroups associated to (d, r, k)-configurations take $M = \gcd(r, k)$, $N = rk \gcd(r, k)$ and $m \leq q \gcd(r, k)$ for any prime $q \geq \max(r, k)$.

What about other linear patterns on $S_{(r,k)}$?

- Addition is a pattern...
- Some patterns are admitted by $S_{(r,k)}$ for special parameters.
  A numerical semigroup admitting the pattern $X+1$ is of the form $\{0, m, m+1, m+2, \dots\}$ and is called **ordinary**, meaning **multiplicity=conductor**.
  We know that $S_{(r,k)}$ is ordinary for

r	3	4	4
k	x	4	5

  We know that $S_{(r,k)}$ is **not** ordinary for

r	5	6
k	5	6

  What about $(r,k)=(5,6)$? We know that multiplicity is 5 and conductor at most 7, but is $6 \in S_{(5,6)}$?