Prime ideals in rings of power series and polynomials

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(work of E. Celikbas, C. Eubanks-Turner, SW)

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Honolulu 2019
For $R$ a commutative ring, $\text{Spec}(R) := \{\text{prime ideals of } R\}$, a partially ordered set under $\subseteq$. 

Questions.

Q1. Work of Nagata '50s; Hochster '69; Lewis and Ohm '71(?), McAdam '77, Heitmann '77,'79; Ratliff '60s-70s

Q2. What posets arise as $\text{Spec}(R)$ for $R$ a 2-dim Noetherian domain? What is $\text{Spec}(R)$ for a particular ring $R$? i.e. Give a characterization of that poset? e.g. $R$ a polynomial ring? Or a ring of power series? [Work of R. Wiegand, Heinzer, S. Wiegand, A.Li, Saydam 70s-90s]

Or a ring that has both?

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Prime ideals, power series
Intro, history:

For $R$ a commutative ring, $\text{Spec}(R) := \{\text{prime ideals of } R\}$, a partially ordered set under $\subseteq$.

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Or a ring that has both?
Let $E = k[[x]][y, z]$, $R[[x]][y]$, or $R[y][[x]]$, a mixed poly-power series, where $k = a$ field or $R = a$ 1-dim Noetherian integral domain, Let $Q \in \text{Spec } E$, $\text{ht } Q = 1$, (usually) $Q \neq xE$.

**Goal Question:** What is $\text{Spec}(E/Q)$?

- $\dim(E/Q) \leq 2$. Assume $\dim(E/Q) = 2$. (Dim 1 is easy.)
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- $\dim(E/Q) \leq 2$. Assume $\dim(E/Q) = 2$. (Dim 1 is easy.)

- $E/Q$ catenary, Noetherian.

A ring $A$ is catenary provided for every pair $P \subsetneq Q$ in $\text{Spec}(A)$, the number of prime ideals in every maximal chain of form $P = P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \ldots \subsetneq P_n = Q$ is the same.

**Example:** What is $\text{Spec}(\mathbb{Z}[y][[x]]/(x - \alpha)$, for $\alpha = 2y(y + 1) \in \mathbb{Z}[y]$?
What is \( \text{Spec}(\mathbb{Z}[y][[x]]/(x - \alpha)) \)? Part 1

Here \( E = \mathbb{Z}[y][[x]], R = \mathbb{Z}, \quad Q = (x - \alpha), \quad \alpha = 2y(y + 1) \in \mathbb{Z}[y], \)
\( B := E/Q. \)

Observations:

\( \mathcal{M} \in \text{max } E \& \text{ht } \mathcal{M} = 3 \implies \mathcal{M} = (m, h(y), x), \) where \( m \in \text{max } R, \)
and \( h(y) \) is irreducible in \( (R/m)[y]. \)

\( \implies x \in M, \forall M \in \text{max } B \text{ with ht } M = 2. \)
Here \( E = \mathbb{Z}[y][[x]], \ R = \mathbb{Z}, \ Q = (x - \alpha), \ \alpha = 2y(y + 1) \in \mathbb{Z}[y], \ B := E/Q. \)

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- \( \implies x \in M, \ \forall M \in \text{max } B \text{ with } \text{ht } M = 2. \)

- \( P \in \text{Spec } E, \ x \notin P, \ \text{ht } P = 2 \land P \text{ NON-maximal } \implies P \subseteq \text{UNIQUE ht-3 } \mathcal{M}. \)
What is Spec($\mathbb{Z}[y][[x]]/(x - \alpha)$? Part 1

Here $E = \mathbb{Z}[y][[x]]$, $R = \mathbb{Z}$, $Q = (x - \alpha)$, $\alpha = 2y(y + 1) \in \mathbb{Z}[y]$, $B := E/Q$.

Observations:

• $M \in \text{max } E$ & $\text{ht } M = 3 \implies M = (m, h(y), x)$, where $m \in \text{max } R$, and $h(y)$ is irreducible in $(R/m)[y]$.

• $x \in M$, $\forall M \in \text{max } B$ with $\text{ht } M = 2$.

• $P \in \text{Spec } E$, $x \notin P$, $\text{ht } P = 2$ & $P$ NON-maximal $\implies P \subseteq \text{UNIQUE } \text{ht}-3$ $M$.

• $B_{xB} = \frac{E}{(x,Q)} = \frac{E}{(x,I)} = \frac{R[y]}{l}$, where $l = \{ f(0, y) \mid f(x, y) \in Q \}$.
What is $\text{Spec}(\mathbb{Z}[y][[x]]/(x - \alpha))$? Part 1

Here $E = \mathbb{Z}[y][[x]], R = \mathbb{Z}, \quad Q = (x - \alpha), \quad \alpha = 2y(y + 1) \in \mathbb{Z}[y], \quad B := E/Q$.

Observations:

$\bullet \mathcal{M} \in \max E \& \ \text{ht} \mathcal{M} = 3 \implies \mathcal{M} = (m, h(y), x)$, where $m \in \max R$,
and $\overline{h(y)}$ is irreducible in $(R/m)[y]$.

$\implies x \in M, \ \forall M \in \max B$ with $\text{ht} M = 2$.

$\bullet P \in \text{Spec} E, x \notin P, \ \text{ht} P = 2 \& P$ NON-maximal $\implies P \subseteq \text{UNIQUE ht-3} \mathcal{M}$.

$\bullet \frac{B}{xB} = \frac{E}{(x, Q)} = \frac{E}{(x, l)} = \frac{R[y]}{l}$, where $l = \{f(0, y) \mid f(x, y) \in Q\}$.

These items $\implies$

$\text{Spec} \left( \frac{\mathbb{Z}[y]}{l} \right) = \text{Spec} \left( \frac{\mathbb{Z}[y]}{2y(y+1)} \right)$ is related to $\text{Spec} \left( \frac{\mathbb{Z}[[x]][y]}{(x - \alpha)} \right)$.
\[
\text{Spec } \left( \frac{\mathbb{Z}[y]}{I} \right) = \text{Spec } \left( \frac{\mathbb{Z}[y]}{2y(y+1)} \right)
\]

Previous slide \implies \text{Spec } \left( \frac{\mathbb{Z}[y]}{2y(y+1)} \right) \text{ is part of } \text{Spec } \left( \frac{\mathbb{Z}[x][y]}{(x-2y(y+1))} \right).

\[U_0 = \text{Spec} \left( \frac{\mathbb{Z}[y]}{2y(y+1)} \right)\]
Previous slide $\implies$ \(\text{Spec} \left( \frac{\mathbb{Z}[y]}{2y(y+1)} \right)\) is part of \(\text{Spec} \left( \frac{\mathbb{Z}[x][y]}{(x-2y(y+1))} \right)\).

\(U_0 = \text{Spec} \left( \frac{\mathbb{Z}[y]}{2y(y+1)} \right):\)

\[\mathfrak{n}_0; (y, 5) \in (y)\]

\[\mathfrak{n}_0; (2, y) \in (2)\]

\[\mathfrak{n}_0; (2, y + 1) \in (y + 1)\]

\[\mathfrak{n}_0; (y + 1, 3) \in (y + 1)\]

Let \(F = \{ (y), (2), (y + 1); (2, y), (2, y + 1) \}\) — a sort of skeleton for \(U_0\). We call it the \textit{determinator} set for \(U_0\).
What is Spec($\mathbb{Z}[y][[x]]/(x - \alpha)$? part 2

Here $Q = (x - \alpha), \quad \alpha = 2y(y + 1) \in \mathbb{Z}[y]$.

For $Q \subseteq P \in \text{Spec } E$, let $\overline{P} = \pi(P)$, where $\pi : E \to E/Q$.

Note: Every height-two element has a set of $|\mathbb{R}|$ elements below it and below no other height-two element (not shown).
Theorem: Let $E = R[y][[x]], R[[x]][y]$ or $k[[z]][x, y]; \ |R| \leq |\mathbb{N}|, \ |\max R| = \infty, R = 1$-dim Noetherian domain, or $k$ a field, $|R|, |k| \leq \aleph_0$

Let $Q \in \text{Spec } E$, $\text{ht } Q = 1$, $Q \neq (x)$ with $\text{dim } E/Q = 2$. Let $U = \text{Spec}(E/Q)$, and let $\varepsilon = |\{\text{ht-1 max elements in } U\}|$. Then:
Theorem: Let $E = R[y][[x]], R[[x]][y]$ or $k[[z]][x, y]; \ |R| \leq |\mathbb{N}|,$ \[\max R| = \infty, \ R = 1\text{-dim Noetherian domain, or } k \text{ a field, } |R|, |k| \leq 2_0 \]

Let $Q \in \text{Spec } E, \ \text{ht} \ Q = 1, \ Q \neq (x) \ \text{with} \ \dim E/Q = 2.$

Let $U = \text{Spec}(E/Q)$, and let $\varepsilon = |\{\text{ht}-1 \text{ max elements in } U\}|.$

Then: $\bullet \varepsilon = 0$ or $|R|$, and $\exists F$ finite 1-dim poset $\subset U \setminus \{(0)\}$
that determine $U$ i.e. Every slot outside $F$ and the $\varepsilon$ slot has the same number of elements as for $\mathbb{Z}[y][[x]]/Q$ above.

Notes: 1. $E = R[y][[x]] \implies \varepsilon = 0.$ 2. $E = k[[x]][z, y] \implies \varepsilon = |R|.$

3. For $E = R[[x]][y]$, let $\ell_y(Q) = \{h_t(x) \mid h_t(x)y^t + \cdots h_0(x) \in Q\}$.

Then:

$\ell_y(Q) = (1) \iff \varepsilon = 0; \quad \ell_y(Q) \neq (1) \iff \varepsilon = |R|.$
Properties of $U$ and $F$

What is the set $F$ associated with $U = \text{Spec}(\mathbb{Z}[y][[x]]/Q)$?

$F_0 := \{ \text{non-0, nonmax } j\text{-prime ideals} \} = \{ u \text{ ht-1} \mid |u^\uparrow| \geq 2 \}$.

Also $F_0$ corresponds to $\{ \text{nonmaximal prime ideals of } E \text{ minimal over } (Q, x) \}$ and to $\{ \text{nonmaximal prime ideals of } R[y] \text{ minimal over } l \}$.

$F := (\bigcup_{f \neq g \in F_0} f^\uparrow \cap g^\uparrow) \cup F_0$, a finite set by item 5 below.
Properties of $U$ and $F$

What is the set $F$ associated with $U = \text{Spec}(\mathbb{Z}[y][[x]]/Q)$?

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1. $|\{ \text{ht-2s in } U \}| = |\mathbb{N}|$, $|\{ \text{ht-0s in } U \}| = 1$, $|\{ \text{ht-1s in } U \}| = |\mathbb{R}|$.

2. $\forall t \in U, \text{ ht } t = 2 \implies |t^{\downarrow e}| = |\mathbb{R}|$.

$t^{\downarrow e} = \{ v \in U | v < s \iff s = t \}$.

3. $\bigcup_{f \in F_0} f^\uparrow = \{ \text{ht-2 } \in U \}.$
Properties of $U$ and $F$

What is the set $F$ associated with $U = \text{Spec} (\mathbb{Z}[y][[x]]/Q)$?

$F_0 := \{\text{non-0, nonmax } j\text{-prime ideals}\} = \{u \text{ ht-1 }| |u^\uparrow| \geq 2\}$.

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1. $|\{\text{ht-2s in } U\}| = |\mathbb{N}| \quad |\{\text{ht-0s in } U\}| = 1 \quad |\{\text{ht-1s in } U\}| = |\mathbb{R}|$

2. $\forall t \in U, \text{ht } t = 2 \implies |t^\downarrow^e| = |\mathbb{R}|.$  
   \quad $t^\downarrow^e = \{v \in U \mid v < s \iff s = t\}.$

3. $\bigcup_{f \in F_0} f^\uparrow = \{\text{ht-2 } \in U\}.$

4. $\forall f \in F_0, |f^\uparrow^e| = \mathbb{N}. \quad (\implies F_0 \subseteq \{j\text{-primes}\}).$  
   \quad $f^\uparrow^e = \{u \in U \mid u > t \iff v = t\}.$

5. $\forall f \neq g \in F_0, |f^\uparrow \cap g^\uparrow| < \infty.$
Theorem: For every finite poset $F$ of dim 1, $\exists Q \in \text{Spec}(\mathbb{Z}[y][[x]])$ such that $F$ "determines" $\text{Spec}(\mathbb{Z}[y][[x]]/Q)$.

(technically, want $F$ such that every $ht-1$ element is above 2 $ht-0$ elements of $F$, to ensure distinct $F$’s determine different $U$.)
Existence Theorem

**Theorem:** For every finite poset $F$ of dim 1, $\exists Q \in \text{Spec}(\mathbb{Z}[y][[x]])$ such that $F$ “determines" $\text{Spec}(\mathbb{Z}[y][[x]]/Q)$.

(Technically, want $F$ such that every ht-1 element is above 2 ht-0 elements of $F$, to ensure distinct $F$’s determine different $U$.)
Example: Let $R = \mathbb{Z}_2$ and $I = 2y(2y - 1)(y + 2)\mathbb{Z}_2[y]$. Then $\text{Spec}(\mathbb{Z}_2[y]/(2y(2y - 1)(y + 2)\mathbb{Z}_2[y]))$ is shown below:

\[
\begin{array}{c}
\aleph_0 \\
(2y - 1) \\
(2) \\
(y) \\
(y + 2)
\end{array}
\]

Spec \left( \frac{\mathbb{Z}_2[y]}{2y(2y-1)(y+2)\mathbb{Z}_2[y]} \right)

The structure of $\text{Spec}(\mathbb{Z}_2[y]/(2y(2y - 1)(y + 2)\mathbb{Z}_2[y]))$ is determined by the partially ordered sets

\[ F = \{(2), (y), (y + 2), (2, y)\} \quad \text{and} \quad G = \{(2y - 1)\}, \]

and by the cardinalities $|(2)^\uparrow e| = \aleph_0$, and $|(y)^\uparrow e| = |(y + 2)^\uparrow e| = 0$. Then $|\min(F)| = 3$, $|G| = 1$ and the type is $(1; F; \aleph_0, 0, 0)$. 

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Example: Let $E = \mathbb{Z}_2[y][x]$, $Q = (x - 2y(2y - 1)(y + 2))E$, and $B = \mathbb{Z}_2[y][x]/Q$. Spec $B$ is shown below except $\exists$ boxes of size $|\mathbb{R}|$ under every height-two element in the box of $\aleph_0$ elements. As above, $I = 2y(2y - 1)(y + 2)\mathbb{Z}_2[y]$.

Spec $B$ for $B = \frac{\mathbb{Z}_2[y][x]}{(x - y(y-2)(y-3)(2y+1)(4y-1))}$
THANKS!