Prime ideals in rings of power series and polynomials

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a partially ordered set under $\subseteq$.

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Or a ring that has both?

Setting/Goal

Let $E = k[[x]][y,z]$, $R[[x]][y]$, or $R[y][[x]]$, a mixed poly-power series, where $k =$ a field or $R =$ a 1-dim Noetherian integral domain,

Let $Q \in \operatorname{Spec} E$, $\operatorname{ht} Q = 1$, (usually) $Q \neq xE$.

Goal Question: What is $\operatorname{Spec}(E/Q)$?

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• E/Q catenary, Noetherian.

A ring A is catenary provided for every pair $P \subsetneq Q$ in $\operatorname{Spec}(A)$, the number of prime ideals in every maximal chain of form

$$P = P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \ldots \subsetneq P_n = Q \text{ is the same.}$$

Example: What is $\operatorname{Spec}(\mathbb{Z}[y][[x]]/(x - \alpha)$, for $\alpha = 2y(y + 1) \in \mathbb{Z}[y]$?

What is Spec$(\mathbb{Z}[y][[x]]/(x-\alpha)$? Part 1

Here $E = \mathbb{Z}[y][[x]], R = \mathbb{Z}, \quad Q = (x - \alpha), \quad \alpha = 2y(y+1) \in \mathbb{Z}[y]$,
B:=E/Q.

Observations:

$\bullet \mathcal{M} \in \max E$ & $\operatorname{ht} \mathcal{M} = 3 \implies \mathcal{M} = (\mathfrak{m}, h(y), x)$, where $\mathfrak{m} \in \max R$, and $\overline{h(y)}$ is irreducible in $(R/m)[y]$.

$\implies x \in M, \ \forall M \in \max B$ with $\operatorname{ht} M = 2$.

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These items $\implies$

$$\operatorname{Spec}\left(\frac{\mathbb{Z}[y]}{I}\right) = \operatorname{Spec}\left(\frac{\mathbb{Z}[y]}{2y(y+1)}\right) \text{ is related to } \operatorname{Spec}\left(\frac{\mathbb{Z}[[x]][y]}{(x-\alpha)}\right).$$

Spec $(\frac{\mathbb{Z}[y]}{I})$= Spec $(\frac{\mathbb{Z}[y]}{2y(y+1)})$

Previous slide $\implies$ Spec $(\frac{\mathbb{Z}[y]}{2y(y+1)})$ is part of Spec $(\frac{\mathbb{Z}[[x]][y]}{(x-2y(y+1))})$.

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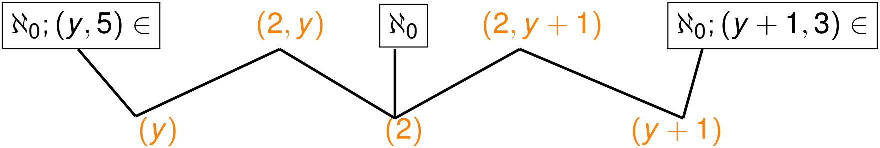


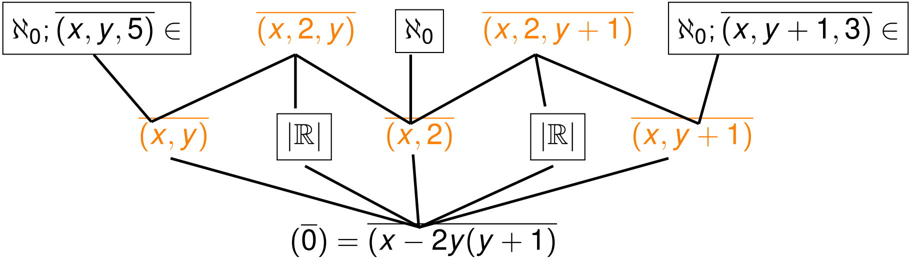
Let $F = \{(y), (2), (y+1); (2, y), (2, y+1)\}$—a sort of skeleton for U_0.

We call it the *determinator* set for U_0.

What is Spec($\mathbb{Z}[y][[x]]/(x-\alpha)$)? part 2

Here $\quad Q = (x - \alpha), \quad \alpha = 2y(y+1) \in \mathbb{Z}[y]$

For $Q \subseteq P \in \operatorname{Spec} E$, let $\overline{P} = \pi(P)$, where $\pi : E \to E/Q$.



Note: *Every* height-two element has a set of $|\mathbb{R}|$ elements below it and below no other height-two element (not shown).

$U = \mathrm{Spec}(R[y][[x]]/Q)$ if $|R| \le \aleph_0$, $|\max R| = \infty$.

Theorem: Let $E = R[y][[x]], R[[x]][y]$ or $k[[z]][x, y]$; $|R| \le |\mathbb{N}|$, $|\max R| = \infty$, $R =$ 1-dim Noetherian domain, or k a field, $|R|, |k| \le \aleph_0$

Let $Q \in \mathrm{Spec}\, E$, ht $Q = 1$, $Q \neq (x)$ with dim $E/Q = 2$.
Let $U = \mathrm{Spec}(E/Q)$, and let $\varepsilon = |\{\text{ht-1 max elements in } U\}|$.

Then:

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Let $U = \mathrm{Spec}(E/Q)$, and let $\varepsilon = |\{$ht-1 max elements in $U\}|$.

Then: $\bullet \varepsilon = 0$ or $|\mathbb{R}|$, and $\bullet \exists F$ finite 1-dim poset $\subset U \setminus \{(0)\}$

that determine U i.e. Every slot outside F and the ε slot has the same number of elements as for $\mathbb{Z}[y][[x]]/Q$ above.

Notes: 1. $E = R[y][[x]] \implies \varepsilon = 0$. 2. $E = k[[x]][z, y] \implies \varepsilon = |\mathbb{R}|$.

3. For $E = R[[x]][y]$, let $\ell_y(Q) = \{h_t(x) \mid h_t(x)y^t + \cdots h_0(x) \in Q\}$.
Then:

$$\ell_y(Q) = (1) \iff \varepsilon = 0; \qquad \ell_y(Q) \neq (1) \iff \varepsilon = |\mathbb{R}|.$$

What is the set F associated with $U = \text{Spec}(\mathbb{Z}[y][[x]]/Q)$?

$F_0 := \{$ non-0, nonmax j-prime ideals$\} = \{u \text{ ht-1} \mid |u^{\uparrow}| \geq 2\}$.

Also F_0 corresponds to { nonmaximal prime ideals of E minimal over (Q, x)} and to {nonmaximal prime ideals of $R[y]$ minimal over I }.

$F := (\bigcup_{f \neq g \in F_0} f^{\uparrow} \cap g^{\uparrow}) \cup F_0$, a finite set by item 5 below.

Properties of U and F

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2. $\forall t \in U, \text{ht}\, t = 2 \implies |t^{\downarrow e}| = |\mathbb{R}|$.
   $t^{\downarrow e} = \{v \in U \mid v < s \iff s = t\}$.
3. $\bigcup_{f \in F_0} f^{\uparrow} = \{\text{ht-2} \in U\}$.

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4. $\forall f \in F_0, |f^{\uparrow e}| = \mathbb{N}$. $(\implies F_0 \subseteq \{j\text{-primes}\}.)$
   $f^{\uparrow e} = \{u \in U \mid u > t \iff v = t\}$.
5. $\forall f \neq g \in F_0, |f^{\uparrow} \cap g^{\uparrow}| < \infty$.

Existence Theorem

Theorem: For every finite poset F of dim 1, $\exists Q \in \text{Spec}(Z[y][[x]])$ such that F “determines" $\text{Spec}(\mathbb{Z}[y][[x]]/Q)$.

(Technically, want F such that every ht-1 element is above 2 ht-0 elements of F, to ensure distinct F’s determine different U .)

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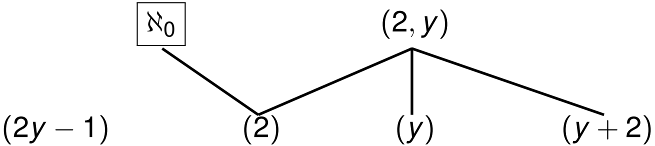
Other examples of spectra, 1

Example: Let $R = \mathbb{Z}_{(2)}$ and $I = 2y(2y-1)(y+2)\mathbb{Z}_{(2)}[y]$. Then Spec($\mathbb{Z}_{(2)}[y]/(2y(2y-1)\,(y+2)\mathbb{Z}_{(2)}[y])$) is shown below:



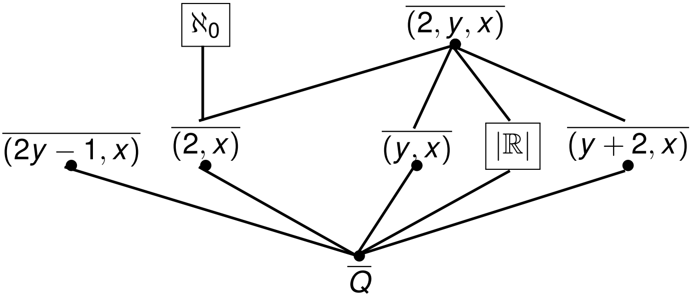
$$\operatorname{Spec}\left(\frac{\mathbb{Z}_{(2)}[y]}{2y(2y-1)(y+2)\mathbb{Z}_{(2)}[y]}\right)$$

The structure of Spec($\mathbb{Z}_{(2)}[y]/(2y(2y-1)(y+2)\mathbb{Z}_{(2)}[y])$) is determined by the partially ordered sets

$$F = \{(2), (y), (y+2), (2, y)\} \quad \text{and} \quad G = \{(2y-1)\},$$

and by the cardinalities $|(2)^{\uparrow e}| = \aleph_0$, and $|(y)^{\uparrow e}| = |(y+2)^{\uparrow e}| = 0$. Then $|\min(F)| = 3$, $|G| = 1$ and the *type* is $(1; F; \aleph_0, 0, 0)$.

Example: Let $E = \mathbb{Z}_{(2)}[y][\![x]\!]$, $Q = (x - 2y(2y-1)(y+2))E$, and $B = \mathbb{Z}_{(2)}[y][\![x]\!]/Q$. Spec B is shown below except $\exists$ boxes of size $|\mathbb{R}|$ under every height-two element in the box of $\aleph_0$ elements. As above, $I = 2y(2y-1)(y+2)\mathbb{Z}_{(2)}[y]$.



Spec B for $B = \dfrac{\mathbb{Z}_{(2)}[y][\![x]\!]}{(x-y(y-2)(y-3)(2y+1)(4y-1))}$

THANKS!