RESEARCH STATEMENT

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My research involves combinatorial aspects of semigroup theory and discrete geometry, with an emphasis on computation, algorithms, and the development and use of software in mathematical research. More specifically, my research goals are to

- (1) develop and implement open source software for use in algebraic statistics and semigroup theory,
- (2) examine the asymptotic behavior of combinatorial structures arising from affine semigroups, and
- (3) characterize expected invariant behavior for randomly sampled semigroups.

Computer software plays a prominent role in my research, and Section 1 details some of the software packages to which I have contributed. The remaining sections focus on my research in semigroup theory. In Section 2, fundamental tools from combinatorial commutative algebra are introduced, several specific invariants of interest in discrete optimization are defined, and specific research directions concerning their asymptotic behavior are proposed. Finally, Section 3 discusses "random semigroups" and a novel approach to a longstanding conjecture in the numerical semi-groups literature.

Undergraduate research. Suitably chosen research problems in semigroup theory are ideal for undergraduate projects, as they offer a high probability of producing publishable output while remaining accessible to students. While significant contributions to open problems in commutative algebra often require extensive background in the subject, semigroup theory can place algebraic objects in combinatorial settings, making them more concrete to work with explicitly. In fact, several of the motivating results presented below are products of undergraduate research projects I have advised, either in the REU setting [CCMMP14, CGHORWW17, CGP14, HKMOP14, KOP15, OPTW15] or independently [BOP14a, BOP14b, GOW17, HO17a, LO18], as are several of the resulting open-source software packages [CO18, DGM17, HO17b, KOS17, OS18, OZ18].

Many of the problems presented below are excellent starting places for undergraduate research projects. As an example, initial investigations for Projects 3 and 4 will utilize mathematics software packages to examine invariant values, providing a natural starting place for students to acquire familiarity and build intuition. Additionally, improvements on the computation in Table 1 and the software packages proposed in Projects 1 and 2 are ideal for computer science students interested in computational mathematics.

1. Software packages

1.1. Algebraic statistics and the m2r package. Algebraic statistics, defined broadly as the application of commutative algebra and algebraic geometry to statistical problems, is generally understood to include applications of other mathematical fields that have substantial overlap with commutative algebra and algebraic geometry, such as combinatorics, polyhedral geometry, graph theory, and others [DSS09, AHT12]. Now a quarter century old, algebraic statistics exploits the recognition that many statistical objects are or can be identified with geometric objects amenable to algebraic investigation, a viewpoint that has been successfully utilized used in discrete multivariate analysis, discrete and Gaussian graphical models, statistical disclosure limitation, phylogenetics, Bayesian statistics, and more. While the field is active and visibly growing, advances in statistical methods made possible by algebraic statisticians are still not mainstream among applied statisticians, largely due to the lack of algebraic algorithms in mainstream statistical software.

Macaulay2 is a popular open source computer algebra system that performs computations in commutative algebra and algebraic geometry [M2]. Well-known for its efficiency with large algebraic computations, the software has a large code base with many community members actively developing add-on packages. In addition, Macaulay2 links to other major open source software in the mathematics community, such as Normaliz [BI10, BIS16], 4ti2 [4TI2], and PHCpack [Ver99], through a variety of interfaces [GPV13, BK10].

R is increasingly the lingua franca of statistics and open source data analytics [RCT14], though it has little to no native support for symbolic computing. For years, researchers in algebraic statistics have been forced to go outside of R to manually run key algebraic computations in software such as Macaulay2 and then pull the results back into R, an error prone and tedious process that presents a real barrier to entry to those wanting to apply algebraic statistics. The problem is compounded by researchers needing to be familiar with the Macaulay2 language, which is syntactically very different from R.

To remedy this situation, David Kahle, Jeff Sommars, and I developed m2r [KOS17], a software package that eliminates many of these difficulties by providing a convenient and intiutive interface between R and Macaulay2, allowing the R user to run Macaulay2 computations without leaving R. More specifically, m2r connects R to a persistent local or remote Macaulay2 session and leverages the mpoly package's existing infrastructure [Kah13] to provide wrappers for commonly used algebraic algorithms in a way that naturally fits into the R ecosystem, alleviating the need to learn Macaulay2. Initially developed at the 2016 Mathematics Research Communities on Algebraic Statistics, the m2r package continues to generate excitement in the field.

One of the primary goals of the m2r package is to lower the barrier to entry for statisticians wishing to use algebraic statistics in their computations. In addition to the theoretical overhead of learning algebraic geometry and the tedious task of learning Macaulay2 syntax, one of the biggest difficulty for users is installing Macaulay2 on their personal machines, an especially arduous tasks for Windows users in particular. The newest version of m2r circumvents the need for users to install Macaulay2 by providing a server version of m2r, enabling m2r users to utilize a remote machine already running Macaulay2 via the internet.

One of the primary remaining tasks in the development of m2r is to increase support for Macaulay2 functionality. Although several of the critical Macaulay2 types are currently supported, there are still many types for which support is needed. Built into the backend of m2r is a parser designed to be as extensible as possible, so that new features can be added easily and quickly. Most of the inner workings of the parser are also black-boxed, so that adding new features does not require a deep understanding of the parser's internal code structure. Thanks to this highly extensible package core, it will be possible to quickly incorporate new features into m2r, thereby greatly increasing its usability to statisticians.

Project 1. Add support for an arsenal of standard Macaulay2 types in the m2r package.

1.2. Computation in semigroup theory. Affine semigroups (finitely generated, additive subsemigroups of $\mathbb{Z}_{\geq 0}^d$) have played an increasingly important role in semigroup theory in recent years, due in large part to their effective use in algebraic and geometric settings involving combinatorics, in addition to semigroup theory itself. Indeed, many landmark results in semigroup theory center around exhibiting extremal behavior, and the setting of affine semigroups is broad enough to encompass a large variety of such behavior [ACHP07, CK15, OPe17a]. At the same time, affine semigroups benefit from algorithms and implementations for explicit computation, which have been instrumental in their study [DGM17, Gar15]. As such, affine semigroups provide a wealth of explicit examples, and obtaining a more thorough understanding of their structure has implications on the general setting.

Non-negative integer solutions to linear equations, which are central to discrete optimization and have a multitude of applications across nearly every discipline [EFRS06, Lee08, LLS08], are



FIGURE 1. An affine semigroup in $\mathbb{Z}_{\geq 0}^2$ with its 7 minimal generators in red (left), and the non-negative integer solutions to $n = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3$ that each represent a factorizations of n in the numerical semigroup $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \subset \mathbb{Z}_{\geq 0}$ (right).

intimately tied to affine semigroups. One of the central themes in semigroup theory are expressions of semigroup elements as sums of generators, called *factorizations*, and for an affine semigroup $S = \langle \alpha_1, \ldots, \alpha_k \rangle \subset \mathbb{Z}_{\geq 0}^d$, the factorizations of a given element $\alpha \in S$ coincide with non-negative integer solutions to a system of linear equations. More specifically, the *set of factorizations* of α , given by

$$\mathsf{Z}_{S}(\alpha) = \{(x_{1}, \dots, x_{k}) \in \mathbb{Z}_{\geq 0}^{k} : \alpha = x_{1}\alpha_{1} + \dots + x_{k}\alpha_{k}\}$$

is precisely the set of non-negative integer solutions to the equation $Ax = \alpha$, where the matrix $A \in \mathbb{Z}^{d \times k}$ has columns $\alpha_1, \ldots, \alpha_k$. This correspondence goes both ways; in any setting involving non-negative integer solutions to a linear system $Ax = \alpha$, an affine semigroup structure is present, and this viewpoint has proven fruitful on countless occasions.

Additionally, affine semigroups arise in commutative algebra in the presence of gradings, which assign to each monomial x^a in the polynomial ring $R = \Bbbk[x_1, \ldots, x_k]$ with coefficient field \Bbbk an element of the semigroup S as its graded degree [MS05, Sta96]. The simplest example is the "standard grading" where $S = \mathbb{Z}_{\geq 0}$ and the graded degree of each monomial is its total degree. Under minimal assumptions, the algebraic structure of R is then determined (up to finite dimensional linear transformations) by the additive semigroup structure of S, and many quantities of interest in commutative algebra and algebraic geometry can be expressed in terms of S, resulting in combinatorial formulas [Hoc77] and algorithms for explicit computation [BS98, HM05]. Additionally, relations between affine semigroup generators correspond to toric ideal generators, the simultaneous solutions of which arise in countless applications [DS98].

The special case of numerical semigroups (additive subsemigroups of $\mathbb{Z}_{\geq 0}$) also plays a unique role in additive combinatorics. Numerical semigroups are central to the Frobenius coin-exchange problem, which ask for the largest non-negative integer value that cannot be evenly changed using a collection of relatively prime coin values n_1, \ldots, n_k [Alf05, BR07]. In this context, each coin value represents an irreducible element of the numerical semigroup $S = \langle n_1, \ldots, n_k \rangle \subset \mathbb{Z}_{\geq 0}$, and factorizations of an element $m \in S$ correspond to distinct ways of making change for m.

Combinatorially flavored invariants, which assign a value to each semigroup element based on its factorizations, play a major role in the study of affine semigroups in each of the aforementioned areas, as they provide a concrete measure of the quantity and distribution of factorizations within a given semigroup. For example, one could choose the optimal value of some quantity (e.g. a linear functional) defined on the level of factorizations. This task arises frequently in linear programming when searching for an optimal non-negative integer solution of a linear system [PS82]. Invariants from combinatorial commutative algebra are often closely connected to affine semigroup structure as well. For instance, minimal relations between semigroup generators can be used to characterize the Betti numbers of certain graded polynomial modules [BPS98].

Discovery of many results on combinatorial invariants has relied on effectively utilizing computer algebra packages, such as the popular GAP package numericalsgps [DGM17]. Analyzing the asymptotic behavior of these invariants often requires computing the invariant for numerous semigroup elements, a task that quickly becomes computationally infeasible for semigroups with many generators due to the number of factorizations that must be computed in each individual computation before passing to the invariant of interest. Recent joint work utilizes dynamic programming to compute invariant values in quick succession without first computing the set of factorizations [BOP14b, GOW17], thereby significantly improving runtime and memory usage.

Theorem 1.1 ([BOP14b]). Given an affine semigroup element $m \in S$, there are dynamic algorithms to compute the sets of factorizations (along with several invariants derived from them) of all divisors of m that outperform standard element-by-element algorithms.

The above algorithms are implemented for numerical semigroups in the popular GAP package numericalsgps [DGM17], for which a Sage wrapper NumericalSemigroup.sage is also available. The algorithms discussed in Theorem 1.1 work for more generally for any affine semigroup, and appear in the recently developed Sage package [CO18].

Project 2. Utilize the new Sage package affinesgps-sage to investigate the asymptotic behavior of combinatorial invariants over affine semigroups (see Theorem 2.2).

2. Combinatorial invariants and semigroup theory

2.1. Quasipolynomials and Hilbert's theorem. The length of a factorization f of a semigroup element $m \in S$ is the number of generators in f; L(m) denotes the set of factorization lengths of m (its length set). Many invariants of interest in semigroup theory are derived from factorization lengths, the simplest and most natural of which are maximum and minimum factorization length, which assign to each semigroup element the values $M(m) = \max L(m)$ and $\mathfrak{m}(m) = \min L(m)$, respectively. Maximum and minimum factorization length also arise in integer programming problems, where factorizations in a given affine semigroup coincide with integer solutions to a set of linear equations [CGLR06, DeL05, DDK13]. In this setting, a semigroup element's factorizations that achieve the maximum (resp. minimum) factorization length are precisely those solutions with maximal (resp. minimal) ℓ_1 -norm.

Recent joint work with Barron and Pelayo has uncovered explicit characterizations in the special case of numerical semigroups for the eventual behavior of maximum and minimum factorization length invariants [BOP14a]. In particular, the maximum factorization length invariant over any numerical semigroup $S = \langle n_1, \ldots, n_k \rangle$ satisfies

$$\mathsf{M}(m) = \frac{1}{n_1}m + a_0(m)$$

for all $m > n_1 n_k$, where a_0 is some n_1 -periodic function depending on S. The above function is an example of a *quasipolynomial*, that is, a polynomial with periodic coefficients.

This characterization relies on the fact that numerical semigroups (i) possess only finitely many irreducible elements and (ii) can be totally ordered. Recently, I have successfully generalized this result from numerical semigroups to affine semigroups, using techniques from combinatorial commutative algebra [One15]. As affine semigroups need not possess a total ordering, these extensions are stated using an appropriate multivariate quasipolynomial analog.

Definition 2.1. A function $f : \mathbb{Z}_{\geq 0}^d \to \mathbb{R}$ is eventually quasipolynomial if (i) f restricts to a polynomial on each of finitely many cones (that is, sets of the form

$$C(\beta;\alpha_1,\ldots,\alpha_r) = \beta + \alpha_1 \mathbb{Z}_{\geq 0} + \cdots + \alpha_r \mathbb{Z}_{\geq 0}$$

for β and linearly independent $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}_{\geq 0}^d$), and (ii) the union of these cones equals $\mathbb{Z}_{\geq 0}^d$.

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Hilbert's theorem, a cornerstone of commutative algebra, states that the Hilbert function of any finitely generated, positively multigraded module is eventually quasipolynomial [Sta96, Fie00]. Hilbert's theorem is often applied to problems in enumerative combinatorics by constructing a graded module whose Hilbert function coincides with a counting function of interest [Ehr62, Mac71, DLMO09]. For example, constructions of this type yield an alternative proof that the number of proper k-colorings of a finite undirected graph G is a polynomial in k whose degree is the number of vertices of G [Sta12].

My recent work characterizes the eventual behavior of maximum and minimum factorization length for any affine semigroup by applying Hilbert's theorem in precisely this manner [One15]. More specifically, given an affine semigroup $S \subset \mathbb{Z}_{\geq 0}^d$, a family of multigraded modules is constructed whose Hilbert functions determine the maximum factorization length of each element of S. Applying Hilbert's theorem to the modules in this family yields a description of the factorization invariant in terms of multivariate eventually quasipolynomial functions. When S is a numerical semigroup (that is, when d = 1), this specializes to a known result for each invariant.

Theorem 2.2 ([One15]). The max and min factorization length functions $M, m : S \to \mathbb{Z}_{>0}$ are eventually quasilinear for any affine semigroup $S \subset \mathbb{Z}_{\geq 0}^d$. Moreover, if $S = \langle n_1, \ldots, n_k \rangle \subset \mathbb{Z}_{\geq 0}$ is a numerical semigroup, then

$$M(m) = \frac{1}{n_1}m + a_0(m)$$
 and $m(m) = \frac{1}{n_k}m + b_0(m)$

for $m \ge n_{k-1}n_k$, where a_0 and b_0 are n_1 -periodic and n_k -periodic functions, respectively.

Theorem 2.2 states that the function $M : S \to \mathbb{Z}_{>0}$ over any affine semigroup S coincides with an eventual quasipolynomial in which the degree of each polynomial restriction is at most 1. If $S = \langle n_1, \ldots, n_k \rangle$ is a numerical semigroup, then more can be said. In particular, the polynomial restrictions occur on n_1 1-dimensional cones, each generated by n_1 , whose union has finite complement in S. The restriction of M to each 1-dimensional cone is linear, and the leading coefficient of each linear restriction is $\frac{1}{n_1}$. The remaining cones are all 0-dimensional, one for each element of Soutside of the 1-dimensional cones.

Theorem 2.2 is part of a larger collection of my results from semigroup theory. Several purely semigroup-theoretic invariants, including delta sets [BCKR06], ω -primality [OPe14], and catenary degree [BGG11], extract highly specialized information about the semigroup's factorization structure, and each has been recently shown to be eventually periodic or quasilinear for any numerical semigroup S [CHK09, OPe13, CCMMP14, OPe15]. My work in [One15] simulataneously generalizes these results to affine semigroups using Hilbert's theorem in the same manner as Theorem 2.2, resulting in a definitive link between combinatorial commutative algebra and invariants of interest in semigroup theory, as well as a new framework through which semigroup theorists can examine these invariants for semigroups with finitely many generators. This illustrates a crutial benefit of using Hilbert's theorem: bringing new tools to the table in adjacent areas.

2.2. Norm-optimizing factorizations. Theorem 2.2 implies that the extremal ℓ_1 -norms of the factorizations of sufficiently large affine semigroup elements $m \in S$ are parametrized by the extremal ℓ_1 -norms of their divisors. Moreover, the proof of Theorem 2.2 for numerical semigroups characterizes which factorizations achieve such values, yielding an explicit parametrization ideal for use in discrete optimization problems.

The ℓ_1 -norm is a special case of a more general class of norms given by linear functionals. In particular, given a weight vector $w \in \mathbb{Z}_{\geq 1}^k$, consider the norm $||(a_1, \ldots, a_k)||_w = w_1 a_1 + \cdots + w_k a_k$, which clearly specializes to the standard ℓ_1 -norm when $w = (1, \ldots, 1)$. Results in discrete geometry involving the ℓ_1 -norm often hold for more general linear functionals, and preliminary computations for several numerical semigroups indicate that Theorem 2.2 is no exception.

Project 3. Prove the following statement.

Given $S \subset \mathbb{Z}_{\geq 0}^d$ affine and $w \in \mathbb{Z}_{\geq 1}^k$, the function sending semigroup elements to the max (resp. min) $\| \cdot \|_w$ -value of their factorizations is eventually quasilinear.

Additionally, express the period and leading coefficient in terms of generators in the case when S is a numerical semigroup, and give a lower bound on the start of periodicity.

The eventually quasipolynomial behavior described in Theorem 2.2 is not limited to linear norms. One such example is the ℓ_0 -norm, which returns the number of nonzero entries in the input vector. A factorization of an affine semigroup element with small 0-norm can be seen as a "sparse solution" to the underlying linear equations. For real solutions, the 0-norm minimization problem has applications in signal processing via compressed sensing, where a linear programming relaxation provides a guaranteed approximation [CERT06, CT05], and random matrices, where upper and lower bounds are known [BDE09, CT05]. In our setting of integer solutions, the 0-norm arises in the study of error-correcting codes, where it coincides with Hamming distance, and 0-norm minimization arises as the nearest codeword problem [APY09, Mic14, Var97]. Additionally, the 0-norm minimization problem arises in the context of finding guarantees for bin-packing problems via the Gilmore-Gomory formulation [GG61, KK82].

Recent joint work with Aliev, De Loera, and Oertel [ADOO17] proves that the ℓ_0 -norm has eventually quasiconstant minimum over any affine semigroup S. Analogous to Theorem 2.2, this result specializes when S is a numerical semigroups.

Theorem 2.3 ([ADOO17]). For any affine semigroup $S \subset \mathbb{Z}_{\geq 0}^d$, the function $\mathsf{m}_0 : S \to \mathbb{Z}_{>0}$ sending each element $m \in S$ to the minimum 0-norm of its factorizations is eventually quasiconstant. Moreover, if $S = \langle n_1, \ldots, n_k \rangle \subset \mathbb{Z}_{\geq 0}$ is a numerical semigroup, then m_0 is eventually periodic with period lcm (n_1, \ldots, n_k) .

The maximum and minimum values of other norms, such as the ℓ_r -norms for $r \in \mathbb{Z}_{>1}$, are also important in discrete optimization [DHK12, DHK13], and characterizing the eventual behavior of their maximum and minimum values over affine semigroup factorizations would have further impacts in this setting. Since the formula for the ℓ_r -norm of a factorization concludes by taking an *r*-th root, it is unlikely to be eventually quasipolynomial (or even admit a polynomial rate of growth for large semigroup elements). It is equivalent to characterize which factorizations admit maximal or minimal $(\ell_r)^r$ -norm, and this quantity has a better chance of being eventually quasipolynomial since it, unlike the ℓ_r -norm, is a polynomial in the coordinates of the input factorization.

Preliminary computations for large elements in several specific numerical semigroups indicates that $(\ell_2)^2$ -norm values may be eventually quasiquadratic with period equal to the sum of the squares of the generators, providing evidence of a positive answer to Project 4 when r = 2. On the other hand, $(\ell_3)^3$ -norm values in the same semigroups either do not begin quasipolynomial behavior until significantly larger elements, or (more likely) are not eventually quasipolynomal.

Project 4. Given $S \subset \mathbb{Z}_{\geq 0}^d$ affine, determine for which $r \in [2, \infty)$ the function sending semigroup elements to the max (resp. min) $(\ell_r)^r$ -value of their factorizations is eventually quasipolynomial. Specialize any answers in the affirmative to the special case where S is a numerical semigroup.

3. RANDOM NUMERICAL SEMIGROUPS

There is a long history of using randomness and probability to study algebraic objects. One of the earliest examples concerns the expected number of real roots of a polynomial with randomly chosen coefficients [LO38]. The study of random matrices has also spanned the better part of the last century, resulting in beautiful universality theorems [Tao12] and highly efficient randomized algorithms [RK04, DF94]. More recently, properties of random Betti tables [EEL15] were obtained using Boij-Söderberg theory [BS08], chordal networks were paired with probabilistic algorithms to yield significant performance improvements for fundamental algebraic geometry computations [CP17], and there has been a surge in the study of random algebraic objects using tools that have proven successful in studying random combinatorial objects (e.g. graphs [ER59] and simplicial complexes [Kah14]).

There are several reasons for examining the "average behavior" of algebraic objects produced in this fashion. For one, in many cases, the worst-case behavior happens only a small fraction of the time, allowing for randomized or approximate algorithms to be particularly effective. A classical example is Gaussian elimination, which is provably numerically unstable in general [OW82] but is quite stable in practice [TS90]. Additionally, probabilistic arguments can demonstrate the existence of objects with extremal properties when concrete examples are too large or infrequent to construct explicitly. As an example, random flag complexes were recently used to construct families of monomial ideals with asymptotic properties that were previously only conjectured [EY17].

In the realm of numerical semigroups, the Frobenius function F(S) of a "typical" numerical semigroup S was explored by Arnol'd [Arn99] and Bourgain and Sinai [BS07]. More specifically, these papers considered random numerical semigroups S obtained by selecting a generating set uniformly at random from the set

$$\{a \in \mathbb{Z}_{>1}^k : \gcd(a) = 1 \text{ and } \max\{a_1, \dots, a_k\} \le M\}$$

of k-tuples with coordinates at most M, and characterized, among other things, the expected value of the Frobenius number F(S). Further results in this direction can also be found in [AHH11].

3.1. The Erdös-Renyi type model. In recent joint work with De Loera and Wilbourne [DOW17] examines a different model for sampling random numerical semigroups. Under our model, dubbed the *ER-type model* for its resemblance to the Erdös-Rényi model for random graphs [ER59], a random numerical semigroup S is generated according to the following procedure:

- (i) fix an upper bound $M \in \mathbb{Z}_{\geq 1}$ and a probability $p \in [0, 1]$;
- (ii) initialize a set of generators $G = \{0\}$ for S;
- (iii) independently choose with probability p whether to include each $n \leq M$ in G.

ER-type models have been used to generate random graphs, simplicial complexes [Kah14], and monomial ideals [DPSSW17]. Often in these areas, the probability p = p(M) is viewed as a function of M, and asymptotic characterizations of expected behavior as $M \to \infty$ are stated in terms of threshold functions, which delineate substantial changes. For example, if a random graph G with nvertices is chosen by including each edge with probability p (the ER-type model for graphs), then as $n \to \infty$, the probability G is connected approaches 1 if $p = p(n) \gg \log(n)/n$ and approaches 0 if $p(n) \ll \log(n)/n$ [ER59]. We say $\log(n)/n$ is the threshold function for connectedness.

Our main result for random numerical semigroups selected under the ER-type model identifies threshold functions for cofiniteness in $\mathbb{Z}_{\geq 0}$ and finiteness of the expected number of minimal generators e(S), expected number of gaps g(S), and expected Frobenius number F(S) as $M \to \infty$. When each of these quantities remains finite as $M \to \infty$, we give bounds in terms of p.

Theorem 3.1 ([DOW17]). Let S be a random numerical semigroup sampled using the ER-type model with upper bound M and probability p.

(a) If $p(M) \ll 1/M$ as $M \to \infty$, then $S = \{0\}$ asymptotically almost surely (a.a.s.). (b) If $1/M \ll p(M)$ and $p(M) \to 0$ as $M \to \infty$, then S has finite complement in $\mathbb{Z}_{>0}$ a.a.s. and

$$\lim_{M \to \infty} \mathbb{E}[e(S)] = \lim_{M \to \infty} \mathbb{E}[g(S)] = \lim_{M \to \infty} \mathbb{E}[F(S)] = \infty.$$

(c) If $p(M) \gg 0$ as $M \to \infty$, then

$$\lim_{M \to \infty} \mathbb{E}[e(S)], \lim_{M \to \infty} \mathbb{E}[g(S)], \lim_{M \to \infty} \mathbb{E}[F(S)] < \infty.$$

	Lower	Experiments		Upper	Exact
p	bound	M = 25000	M = 50000	bound	M = 90
0.25	2.21	3.3663	3.3761	3.75	2.767
0.1	2.64	4.6236	4.6402	19.9	3.782
0.01	2.96	9.7906	9.776	199.9	0.858

TABLE 1. Comparison of asymptotic bounds on the expected number of minimal generators in Theorem 3.1 with experimental evidence.

All 3 quantities whose asymptotic behavior is characterized in Theorem 3.1 depend closely on the probability $A_n(p)$ that an integer n is not lie in the semigroup generated by the chosen elements less than n. For instance $A_{15}(p) = (1-p)^7(1+4p+2p^2)$ is the probability that 15 cannot be written as a sum of the integers $1, \ldots, 14$ each chosen with probability p. Surprisingly, the form of $A_{15}(p)$ is no coincidence; factoring out $(1-p)^{\lfloor n/2 \rfloor}$ from $A_n(p)$ yields a polynomial in p with non-negative integer coefficients given by the *h*-vector of a simplicial complex Δ_n , that is,

$$A_n(p) = (1-p)^{\lfloor n/2 \rfloor} (h_{n,0} + h_{n,1}p + \cdots).$$

A cornerstone of algebraic combinatorics, the *h*-vector of a simplicial complex Δ is a finite integer sequence determined by the numbers f_i of *i*-dimensional faces of Δ , and arises naturally in the study of squarefree monomial ideals [Sta96] and posets [KO96]. More specifically, the *h*-vector gives the coefficients of the numerator of the Hilbert series of the Stanley-Riesner ring $\Bbbk[\Delta]$ of Δ . If Δ is sufficiently nice (e.g. shellable, Cohen-Macaulay, or partitionable), then the *h*-vector entries are non-negative integers counting certain facets of Δ .

The facets of the simplicial complex Δ_n , whose *h*-vector entries appear in the polynomial $A_n(p)$, are in bijection with those numerical semigroups that are maximal (w.r.t. containment) among numerical semigroups with Frobenius number *n* (called *irreducible* numerical semigroups [RG09]). The complex Δ_n turns out to be shellable, which implies that the *h*-vector entries (and thus the coefficients arising in $A_n(p)$) each count facets of Δ_n with certain properties. In the end, the coefficient $h_{n,i}$ counts the number of irreducible numerical semigroups *S* with F(S) = n and precisely *i* minimal generators less than n/2. Though several posets whose elements are numerical semigroups have been studied elsewhere in the literature [RGGJ03], none examine the simplicial complex Δ_n specifically.

As a consequence of the above discussion, the asymptotic behaviors of $\mathbb{E}[e(S)]$, $\mathbb{E}[g(S)]$ and $\mathbb{E}[F(S)]$ as $M \to \infty$ are controlled via combinatorial bounds on the *h*-vector of the shellable simplicial complex Δ_n . Consequently, parts (b) and (c) of Theorem 3.1 are obtained, as well as upper and lower bounds in terms of *p* when each expectation is finite. Table 1 compares the bounds resulting from the proof of Theorem 3.1 with experimental evidence (100,000 samples), and the last column gives the exact expected value for M = 90 using polynomials computed with the algorithm [RGGJ04] implemented in the popular GAP package numerical sgps [DGM17].

3.2. The probabilistic method and Wilf's conjecture. Wilf's conjecture [Wil78] is one of the most famous open problems in the numerical semigroups literature. Equality has been shown to hold for $M = \langle n_1, n_2 \rangle$ and $M = \langle m, m+1, \ldots, 2m-1 \rangle$, which have respectively the smallest and largest possible number of minimal generators for a numerical semigroup. Aside from a small number of isolated examples, the inequality appears to be strict in all other cases, but despite substantial effort, the conjecture remains open in general.

Conjecture 3.2 (Wilf). Any numerical semigroup $M = \langle n_1, \ldots, n_k \rangle$ satisfies

$$F(M) + 1 \le k(F(M) - g(M)),$$

where F(M) is the Frobenius number of M and g(M) is the number of gaps of M.

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First popularized by Erdös [Erd64], the *probabilistic method* in combinatorics is a method of proving objects with a certain property exist without explicitly producing an example. Often, such proofs involve selecting an object at random and proving the probability of having the desired property is positive [AS16].

The probabilistic method is particularly effective if the objects in question are too large to easily construct but abundant enough to occur frequently. The quintessential example is Erdös proof [Erd63] that the Ramsey numbers R(r,r) grow exponentially in r, which he accomplished by proving that any complete graph with at least $(1.1)^r$ vertices has an edge 2-coloring with no monochromatic r-vertex subgraphs. Rather than explicitly construct such a coloring, he proved that the probability of a randomly chosen edge coloring having this property is positive. To this day, no concrete examples of such a coloring have been found, despite the fact that his proof demonstrates "most" colorings satisfy this property.

Wilf's conjecture is precisely the kind of problem the probabilistic method is designed for. Indeed, every numerical semigroup M with $g(M) \leq 60$ has been shown to satisfy Wilf's conjecture via exhaustive computation [Bra08], so if a counterexample exists, it is not "small". Additionally, many of the special classes of semigroups for which Wilf's conjecture has been proven satisfy some inequality that makes the proof easier (for instance, a lower bound on k [Sam12] or an upper bound on F(M) [DM06]), resulting in a form of "selection bias" favoring classes of semigroups that are more likely to satisfy the conjecture. The probabilistic method provides a way to systematically avoid such bias by considering all numerical semigroups at once, partitioning them based on certain conditions, and ensuring that some collection failing to satisfy Wilf's conjecture is nonempty.

Project 5. Use the probabilistic method to prove a counterexample to Wilf's conjecture exists.

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