Properties of maximum and minimum factorization length in numerical semigroups

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Background Information

- Let *S* be a numerical semigroup with finite complement such that we can write *S* = $<n_1, n_2, \ldots, n_k>$ with $n_i \in \mathbb{N}$, $n_i < n_{i+1}$, and gcd($n_1, n_2, \ldots, n_k$) = 1.
- We define the Apéry Set of *S* with respect to *n* in *S* is:

$$\mathrm{Ap}(S, n) = \{s \in S \mid s-n \notin S\}$$

- Theorem 1: *Let S be a numerical semigroup and let n be a nonzero element of S. Then* AP*(S, n)* = {0 = *w(0), w(1), …, w(n – 1)*}, *where w(i) is the least element of S congruent with i modulo n, for all i in* {0, …, *n* – 1}. [1]

Background (cont.)

- We also have that for sufficiently large n ($n \geq (n_1$ - 1) n_k), the maximum factorization length is quasilinear and can be written as

$$\mathrm{M}(n) = \frac{1}{n_1} n + a(n)$$

for some periodic $a(n)$.

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$$\mathrm{m}(n) = \frac{1}{n_k} n + c(n)$$

for some periodic c(n). [2]

Background (cont.)

Terminology:

- For the purposes of this presentation, a *harmonic* numerical semigroup is one in which for all n,

$$\mathsf{M}(n + n_1) = \mathsf{M}(n) + 1$$

  - Note: We sometimes say *harmonic with respect to minimum length* to refer to the same property but with regards to the minimum length function.
- A *shifted numerical semigroup* is one of the following form:

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle$$

New equations for max and min fact. length

- We can rewrite the equation for maximum factorization length as:

$$\mathrm{M}(n) = \frac{n - b_i}{n_1}$$

for some positive integers $b_i \geq i$ and $i = n \bmod n_1$

- Similarly, we can rewrite the equation for minimum factorization length as:

$$\mathrm{m}(n) = \frac{n + c_j}{n_k}$$

for some positive integers c_j and $j = n \bmod n_k$

- In this presentation we will talk about the derivation of a formula for b_i and c_j

Generalized Definition of the Apéry Set

Suppose that S is a numerical semigroup, not necessarily with finite complement, and $n \in \mathbb{N}$. We define the set

$$\mathrm{Ap}(S, n) = \{m_i \in S \mid for\ 0 \leq i \leq n - 1\}$$

the *Apéry Set* of S with respect to n, where m_i is defined as

$$m_i = \begin{cases} 0, & \text{if } S \cap \{i, i+n, i+2n, \dots\} = \emptyset \\ \min(S \cap \{i, i+n, i+2n, \dots\}, & \text{otherwise} \end{cases}$$

Examples

- Let $S = \langle 6, 9, 20 \rangle$, take Ap$(S, 4)$ and Ap$(S, 6)$

- Let $S = \langle 2 \rangle$, take Ap$(S, 2)$ and Ap$(S, 3)$

Examples

- Let S = $<6, 9, 20>$, take Ap$(S, 4)$ and Ap$(S, 6)$

Ap$(S, 4)$ = $\{0, 9, 6, 15\}$

Ap$(S, 6)$ = $\{0, 49, 20, 9, 40, 29\}$

- Let S = $<2>$, take Ap$(S, 2)$ and Ap$(S, 3)$

Ap$(S, 2)$ = $\{0\}$

Ap$(S, 3)$ = $\{0, 4, 2\}$

Using the new definition to solve our problem

Theorem: Let S be a numerical semigroup with finite complement, such that S = <$n_1, n_2, \ldots, n_k$> for $n_j \in \mathbb{N}$ and $n_j < n_{j+1}$. Take S_{M} to be the numerical semigroup (not necessarily with finite complement) such that

$$S_{\mathrm{M}} = \langle n_2 - n_1, n_3 - n_1, \ldots, n_k - n_1 \rangle$$

then we have that for $n \geq (n_1 - 1)\, n_k$

$$\mathrm{M}(n) = \frac{n - b_i}{n_1}$$

where $b_i \in \mathrm{Ap}(S_{\mathrm{M}}, n_1)$ with $i = b_i \bmod n_1$.

Examples

- Let S = <6, 9, 20>, find the set of b_i of S

- Let S = <9, 10, 21>, find the set of b_i of S

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First we see that $S_{\mathrm{M}} = \langle 3, 14 \rangle$

- Let S = <9, 10, 21>, find the set of b_i of S

First we see that $S_{\mathrm{M}} = \langle 1, 12 \rangle$

Examples

- Let S = <6, 9, 20>, find the set of b_i of S

First we see that $S_{\mathrm{M}} = \langle 3, 14 \rangle$

Now we take $\mathrm{Ap}(S_{\mathrm{M}}, 6)$

- Let S = <9, 10, 21>, find the set of b_i of S

First we see that $S_{\mathrm{M}} = \langle 1, 12 \rangle$

Now we take $\mathrm{Ap}(S_{\mathrm{M}}, 10)$

Examples

- Let S = <6, 9, 20>, find the set of b_i of S

We get that:

$$b_i = \{0, 31, 14, 3, 28, 17\}$$

  - *Note: As it turns out, this S is harmonic*

- Let S = <9, 10, 21>, find the set of b_i of S

We get that:

$$b_i = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$$

  - *Note: As it turns out, this S is NOT harmonic*

Further Examples

- Let *S* = <5, 7>, find the set of b_i of *S*

- Let *S* = <5, 7, 9>, find the set of b_i of *S*

Further Examples

- Let S = <5, 7>, find the set of b_i of S

First we see that $S_{\mathrm{M}} = \langle 2 \rangle$

- Let S = <5, 7, 9>, find the set of b_i of S

First we see that $S_{\mathrm{M}} = \langle 2, 4 \rangle$

Further Examples

- Let S = <5, 7>, find the set of b_i of S

First we see that $S_{\mathrm{M}} = \langle 2 \rangle$

Now we take $\mathrm{Ap}(S_{\mathrm{M}}, 5)$

- Let S = <5, 7, 9>, find the set of b_i of S

First we see that $S_{\mathrm{M}} = \langle 2, 4 \rangle$

Now we take $\mathrm{Ap}(S_{\mathrm{M}}, 5)$

Further Examples

- Let *S* = <5, 7>, find the set of b_i of *S*

We get that:
$b_i = \{0, 6, 2, 8, 4\}$

  - *Note: As it turns out, this S is harmonic*

- Let *S* = <5, 7, 9>, find the set of b_i of *S*

We get that:
$b_i = \{0, 6, 2, 8, 4\}$

  - *Note: As it turns out, this S is harmonic*

Defining the Maximum length Apéry Set

We define the set

$$\mathrm{MAp}(S) = \{b_i + n_1 \cdot \mathrm{m}_{S_{\mathrm{M}}}(b_i) \mid 0 \leq i \leq n_1 - 1\}$$

The Maximum Length Apéry Set of S with respect to n_1, where $\mathrm{m}_{S_{\mathrm{M}}}$ denotes the minimum factorization length in S_{M}.

Key property of the MAp set: The elements $a_i = b_i + n_1 \cdot \mathrm{m}_{S_{\mathrm{M}}}(b_i) \in \mathrm{MAp}(S)$ are the smallest elements in S in each congruence class modulo n_1 such that $\mathrm{M}(a_i) = \frac{a_i - b_i}{n_1}$, that is, $\mathrm{M}(a_i + pn_1) = \mathrm{M}(a_i) + p$ for every $p \geq 0$.

Examples

- Let S = <6, 9, 20>, find the MAp(S)

- Let S = <9, 10, 21>, find the MAp(S)

Examples

- Let $S = \langle 6, 9, 20 \rangle$, find the $\mathrm{MAp}(S)$

$\mathrm{MAp}(S) = \{0, 49, 20, 9, 40, 29\}$

- Let $S = \langle 9, 10, 21 \rangle$, find the $\mathrm{MAp}(S)$

$\mathrm{MAp}(S) = \{0, 10, 20, 30, 40, 50, 60, 70, 90\}$

Further Examples

- Let S = <5, 7>, find the MAp(S)

MAp(S) = {0, 21, 7, 28, 14}

- Let S = <5, 7, 9>, find the MAp(S)

MAp(S) = {0, 16, 7, 23, 9}

Minimum Factorization Length

It turns out that the formula for minimum factorization is very reflexive to the formula for maximum length factorization:

Let S be a numerical semigroup with finite complement, such that S = $<n_1, n_2, \ldots, n_k>$ for $n_j \in \mathbb{N}$ and $n_j < n_{j+1}$. Take S_m to be the numerical semigroup (not necessarily with finite complement) such that

$$S_\mathrm{m} = \langle n_k - n_1, n_k - n_2, \ldots, n_k - n_{k-1} \rangle$$

then we have that for $n \geq (n_k - 1)\, n_k$

$$\mathrm{m}(n) = \frac{n + c_i}{n_1}$$

where $c_i \in \mathrm{Ap}(S_\mathrm{m}, n_k)$ with $\mathrm{c_i} + i = 0 \bmod n_k$.

Defining the Minimum length Apéry Set

We define the set

$$\mathrm{mAp}(S) = \{n_k \cdot \mathrm{m}_{S_\mathrm{m}}(c_i) - c_i \mid 0 \leq i \leq n_k - 1\}$$

The Minimum Length Apéry Set of S with respect to n_k, where $\mathrm{m}_{S_\mathrm{m}}$ denotes the minimum factorization length in S_m.

Key property of the MAp set: The elements $w_i = n_k \cdot \mathrm{m}_{S_\mathrm{m}}(c_i) - c_i \in \mathrm{mAp}(S)$ are the smallest elements in S in each congruence class modulo n_k such that $\mathrm{m}(w_i) = \frac{w_i + c_i}{n_k}$, that is, $\mathrm{m}(w_i + pn_k) = \mathrm{m}(w_i) + p$ for every $p \geq 0$.

Bibliography

[1] Numerical Semigroups, J.C. Rosales, P.A. García-Sánchez

[2] On the set of elasticities in numerical monoids, T. Barron, C. O'Neill, and R. Pelayo

Proof:

Take $n \geq (n_1$ - 1) n_k then we know that we can write n as $n = pn_1 + i$ for some p and for $i = n \bmod \mathrm{n}_1$.

For any factorization of n, $\mathbf{q}$ meaning that we can write

$$n = q_1 n_1 + q_2 n_2 + \cdots + q_k n_k$$

there is a corresponding factorization

$$n - Qn_1 = q_2(n_2 - n_1) + q_3(n_3 - n_1) + \cdots + q_k(n_k - n_1)$$

with $Q = q_1 + q_2 + \ \ldots + q_k$, in S_{M}.

Now, we see that:

$$\max(|\mathbf{q}|) = \max(q_1 + q_2 + \ \ldots + q_k) = \max(Q)$$

So the maximum factorization length occurs for a maximal value of Q.

Proof, cont.

Recall that $n = pn_1 + i$ for some p, so we can rewrite the equation

$$n - Qn_1 = q_2(n_2 - n_1) + q_3(n_3 - n_1) + \cdots + q_k(n_k - n_1)$$

as

$$pn_1 + i - Qn_1 = q_2(n_2 - n_1) + q_3(n_3 - n_1) + \cdots + q_k(n_k - n_1)$$

which simplifies to

$$(p - Q)n_1 + i = q_2(n_2 - n_1) + q_3(n_3 - n_1) + \cdots + q_k(n_k - n_1)$$

We have that $q_j \geq 0$ and $n_j - n_1 \geq 0$ for all j so we must have that the right hand side is greater than or equal to 0. Since $i < n_1$ and the left-hand side is greater than or equal to 0, $p \geq Q$.

Proof, cont.

We have shown that $(p - Q)n_1 + i$ lives in S_{M} and we are now trying to maximize Q. By maximizing Q we are minimizing $(p - Q)n_1 + i$, so in fact, we are looking for the smallest value $b_i \in S_{\mathrm{M}}$ such that $b_i = (p - Q)n_1 + i$, which is the same thing as saying

$$b_i = \min(q_2(n_2 - n_1) + \ldots + q_k(n_k - n_1))$$

Note that this is exactly the same b_i that was calculated using $\mathrm{Ap}(S_{\mathrm{M}}, n_1)$