Binomial Irreducible Decomposition

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Question

Definition

An ideal $I \subset S$ is *irreducible* if whenever $I = J_1 \cap J_2$ for ideals $J_1, J_2 \subset S$, either $I = J_1$ or $I = J_2$.

Definition

An *irreducible decomposition* of an ideal $I \subset S$ is an expression $I = \bigcap_i J_i$ for irreducible ideals $J_i \subset S$.

Question

Definition

An ideal $I \subset \Bbbk[x_1, \ldots, x_n]$ is a *binomial ideal* if it is generated by polynomials of the form $x^a + \lambda x^b$ for $a, b \in \mathbb{N}^n$.

Theorem ([Eisenbud-Sturmfels, 1996])

If $\Bbbk$ is algebraically closed, then every binomial ideal in $\Bbbk[x_1, \ldots, x_n]$ admits a binomial primary decomposition.

Example

$\langle x^4 + 4 \rangle = \langle x^2 - 2x + 2 \rangle \cap \langle x^2 + 2x + 2 \rangle \subset \mathbb{Q}[x]$.

Question (Eisenbud, Sturmfels)

Do binomial ideals have *binomial* irreducible decompositions?

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Question (Eisenbud, Sturmfels)

Do binomial ideals have *binomial* irreducible decompositions?

Answer: Yes, when $\Bbbk$ is algebraically closed (proof in progress).

Special Case: Monomial Ideals

Notice: Bijection between monomials in $\Bbbk[x_1, \ldots, x_n]$ and vectors in $\mathbb{N}^n$.

$$\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n} \longleftrightarrow (a_1, \ldots, a_n) \in \mathbb{N}^n$$

Definition

An ideal $I \subset S$ is a *monomial ideal* if it is generated by monomials.

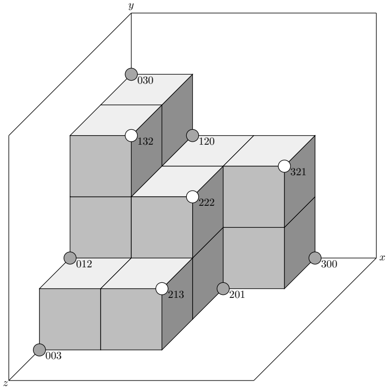
Theorem

Fix a monomial ideal $I \subset S = \Bbbk[x_1, \ldots, x_n]$.

- *I is irreducible if and only if its generators have the form $x_i^{a_i}$.*
- *I admits a monomial irreducible decomposition. Moreover, I admits a unique (up to reordering) irredundant such decomposition.*
- *The irredundant decomposition is independent of $\Bbbk$.*

Special Case: Monomial Ideals

Let $I = \langle x^3, y^3, z^3, xy^2, yz^2, x^2z \rangle \subset \Bbbk[x, y, z]$. Monomials in $\Bbbk[x, y, z]/I$?



$$I = \langle x, y^3, z^2 \rangle \cap \langle x^3, y^2, z \rangle \cap \langle x^2, y, z^3 \rangle \cap \langle x^2, y^2, z^2 \rangle$$

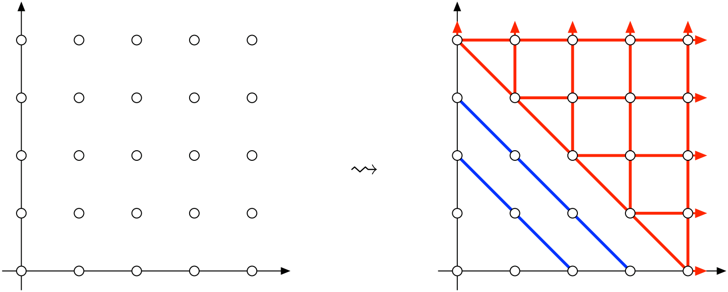
Binomial Ideals

Steps to construct binomial irreducible decompositions:

1. Combinatorial framework (primary reference: [Kahle-Miller, 2013])
2. Condition for irreducibility
3. Method for decomposition

Binomial Ideals

Let $I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \subset \mathbb{k}[x, y]$. Monomials in $\mathbb{k}[x, y]/I$?



$\mathbb{k}[x, y]$ $\rightsquigarrow$ $\mathbb{k}[x, y]/I$

Monoid Congruences

Definition

A *monoid* $(Q, +)$ is a set Q with a binary operation $+$ which is commutative, associative, and has an identity.

Definition

The *monoid algebra* $\Bbbk[Q]$ is the ring consisting of all finite sums of formal elements $\mathbf{x}^a$ for $a \in Q$, with coefficients in $\Bbbk$. Multiplication is given by $\mathbf{x}^a \cdot \mathbf{x}^b = \mathbf{x}^{a+b}$ for $a, b \in Q$.

Example

For $Q = \mathbb{N}^n$, the monoid algebra $\Bbbk[Q]$ is the polynomial ring $\Bbbk[x_1, \ldots, x_n]$.

Monoid Congruences

Definition

A *congruence* $\sim$ on a monoid Q is an equivalence relation which satisfies

$$a \sim b \Rightarrow a + c \sim b + c$$

for all $a, b, c \in Q$. In this case, the set of equivalence classes $\overline{Q} = Q/{\sim}$ forms a monoid.

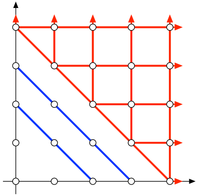
Lemma

Fix a binomial ideal $I \subset \Bbbk[Q]$. The relation $\sim_I$ on Q given by

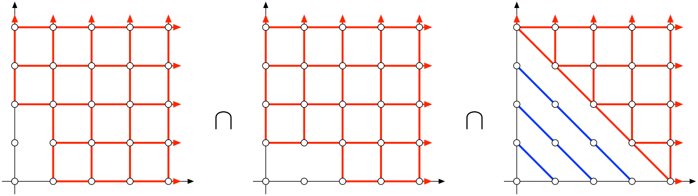
$$a \sim_I b \text{ whenever } \mathbf{x}^a + \lambda \mathbf{x}^b \in I \text{ for some } \lambda \in \Bbbk$$

is a congruence on Q. The distinct monomials in $\Bbbk[Q]/I$ are in bijection with the (non-nil) elements of $\overline{Q} = Q/{\sim_I}$.

Refinement of Congruences



$\langle x^2 - xy, xy - y^2, x^4 \rangle$



$\langle x, y^2 \rangle \quad \cap \quad \langle x^2, y \rangle \quad \cap \quad \langle x - y, x^4 \rangle$

Combinatorial Framework

In review, our combinatorial framework consists of:

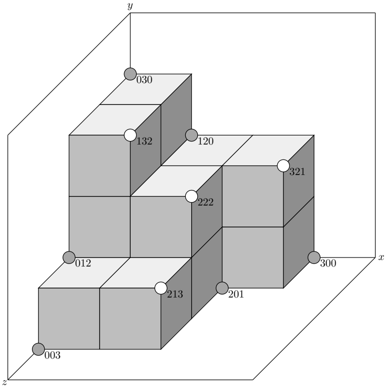
$$\begin{array}{rcl} \text{monomial } \mathbf{x}^a \in \Bbbk[Q] & \longleftrightarrow & \text{element } a \in Q \\ \text{binomial ideal } I \subset \Bbbk[Q] & \longrightarrow & \text{congruence } \sim_I \text{ on } Q \\ \text{intersection } I \cap J \subset \Bbbk[Q] & \longleftrightarrow & \text{refinement } \sim_I \cap \sim_J \text{ on } Q \end{array}$$

We can assume (for the remainder of the talk):

1. $\overline{Q} = Q/\sim$ is finite, and the only cancellative element of $\overline{Q}$ is $\overline{0}$.
2. $\overline{P} = \overline{Q} \setminus \{\overline{0}\}$ is the unique maximal prime ideal.

Key Witnesses

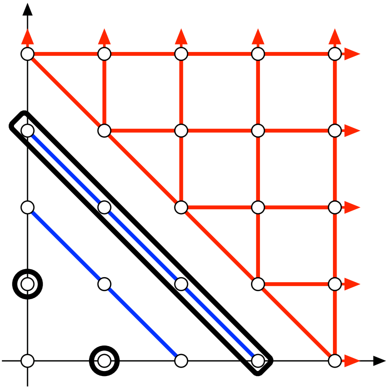
For monomial ideals, "outward corners" $\leftrightarrow$ irreducible components.



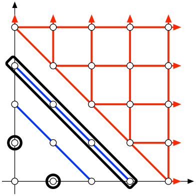
Key Witnesses

Definition

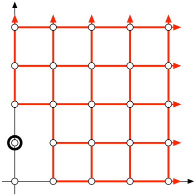
A non-nil $w \in Q$ is a *key witness* for $\sim$ if there exists some $\overline{u} \neq \overline{w}$ with $\overline{w} + \overline{p} = \overline{u} + \overline{p}$ for all $\overline{p} \in \overline{P}$.



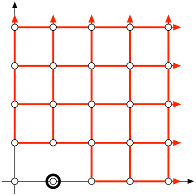
Components at witnesses



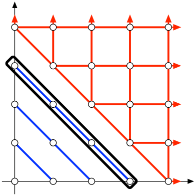
$\langle x^2 - xy, xy - y^2, x^4 \rangle$



$\langle x, y^2 \rangle$



$\langle x^2, y \rangle$



$\langle x - y, x^4 \rangle$

Irreducible Condition

Definition

A congruence $\sim$ on Q is *mesoirreducible* if $Q/\!\sim$ has only one key witness.

Theorem

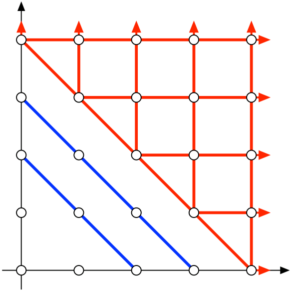
Suppose $\Bbbk = \overline{\Bbbk}$. *If a binomial ideal* $I \subset \Bbbk[Q]$ *induces a mesoirreducible congruence, then it has a canonical binomial primary decomposition, and each component of this decomposition is irreducible.*

Example

$I = \langle x^3 - 1, y \rangle \subset \Bbbk[x, y]$. The congruence $\sim_I$ is mesoirreducible, but
$I = \langle x - 1, y \rangle \cap \langle x^2 + x + 1, y \rangle$.

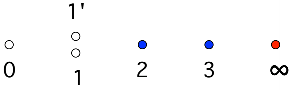
Mesoirreducible Decomposition

$$I = \langle x^2 - xy, xy - y^2, x^4 \rangle$$



$\Bbbk[x, y]/I$

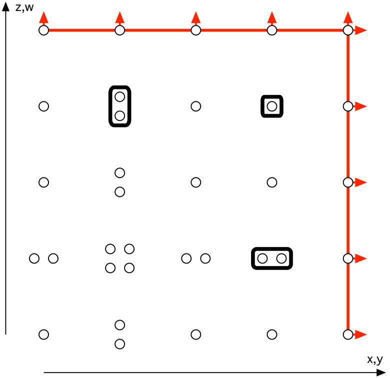
$\rightsquigarrow$



$Q/{\sim_I}$

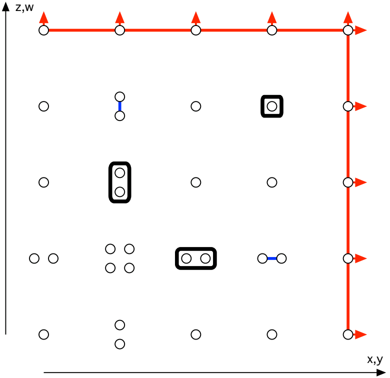
Mesoirreducible Decomposition

$$\langle x^2 - xy, xy - y^2, z^2 - zw, zw - w^2, x^4, z^4 \rangle \subset \Bbbk[x, y, z, w]$$



Mesoirreducible Decomposition

$$\langle x^2 - xy, xy - y^2, z^2 - zw, zw - w^2, x^4, z^4 \rangle \subset \Bbbk[x, y, z, w]$$



Mesoirreducible Decomposition

Definition

An element $w \in Q$ is a *protected witness for* $\sim$ if w becomes a witness after iteration of this process.

Theorem

Any congruence $\sim$ *on* Q *can be expressed as the common refinement of mesoirreducible congruences, one for each protected witness.*

Corollary

If $\mathbb{k} = \overline{\mathbb{k}}$*, then every binomial ideal in* $\mathbb{k}[x_1, \ldots, x_n]$ *admits a binomial irreducible decomposition.*

References

David Eisenbud, Bernd Sturmfels (1996)
Binomial ideals.
Duke Math J. 84 (1996), no. 1, 145.

Ezra Miller, Bernd Sturmfels (2005)
Combinatorial commutative algebra.
Graduate Texts in Mathematics 227. Springer-Verlag, New York, 2005.

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