Irreducible decomposition of binomial ideals

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Joint with Thomas Kahle and Ezra Miller

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The Question

Definition

An ideal $I \subset \Bbbk[x_1, \ldots, x_n]$ is a *binomial ideal* if it is generated by polynomials with at most two terms.

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$\langle x^2 - y, x^2 + y \rangle = \langle x^2, y \rangle \subset \Bbbk[x, y], \langle x^2 y - xy^2, x^3, y^3 \rangle \subset \Bbbk[x, y]$.

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$x^2 - xy, x^3 - x^2, x^4 y^2 + xy^2 \in \langle x^2, y^2, xy \rangle \subset \Bbbk[x, y]$.

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An ideal $I \subset S$ is *irreducible* if whenever $I = J_1 \cap J_2$ for ideals $J_1, J_2 \subset S$, either $I = J_1$ or $I = J_2$.

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Fact

Every ideal $I \subset \mathbb{k}[x_1, \ldots, x_n]$ *can be written as a finite intersection*

$$I = \bigcap_{i=1}^{r} J_i$$

of irreducible ideals $J_1, \ldots, J_r$ *(an irreducible decomposition).*

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Question (Eisenbud-Sturmfels, 1996)

Assume $\Bbbk$ is algebraically closed. Does every binomial ideal I have a *binomial* irreducible decomposition, that is, an expression $I = \bigcap_i J_i$ where each J_i is irreducible and binomial?

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Example

$I = \langle x^2y - xy^2, x^3, y^3 \rangle \subset \Bbbk[x, y]$.

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So, why was this problem was open for almost 20 years?
Answer: Needed to know where to look.

Storytime!

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- Examine the counterexample, with proof (time permitting).

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Example

Primary ideals in $\mathbb{Z}$ are of the form $\langle p^r \rangle$ for p prime, and $\sqrt{\langle p^r \rangle} = \langle p \rangle$.
For $a = p_1^{r_1} \cdots p_\ell^{r_\ell} \in \mathbb{Z}$, $\langle a \rangle = \bigcap_i \langle p_i^{r_i} \rangle$.

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Definition

Given a $\mathfrak{p}$-primary ideal $I \subset \mathbb{k}[x_1, \ldots, x_n]$, the *socle* of I is the ideal

$$\mathrm{soc}_{\mathfrak{p}}(I) = \{f : \mathfrak{p}f \subset I\} \subset I$$

We say I has *simple socle* if $\dim_{\mathbb{k}} \mathrm{soc}_{\mathfrak{p}}(I)/I = 1$.

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Fact

A $\mathfrak{p}$-primary ideal I is irreducible if and only if it has simple socle.

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Let $I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle \subset \Bbbk[x, y]$, and let $\mathfrak{p} = \langle x, y \rangle$.

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so $\dim_{\mathbb{k}}(\mathrm{soc}_{\mathfrak{p}}(I)/I) = 2$.

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Monomial Ideals

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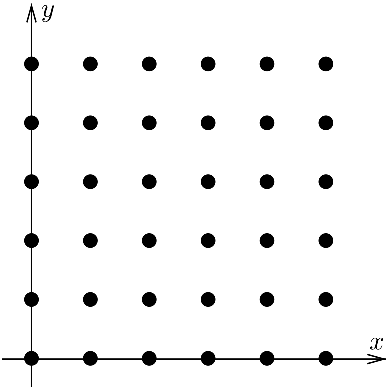

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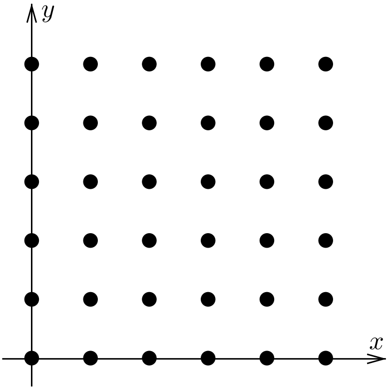

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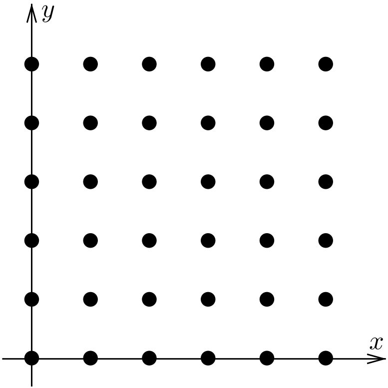

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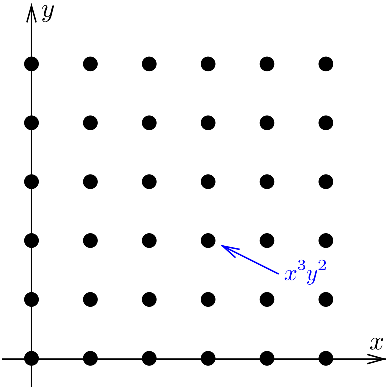

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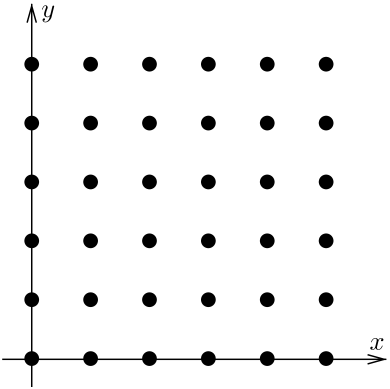

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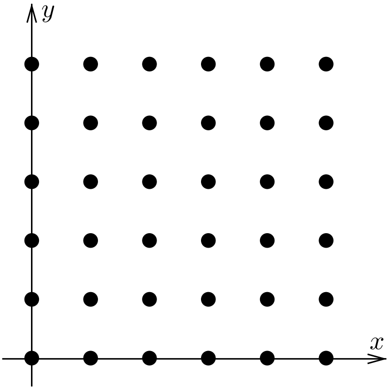

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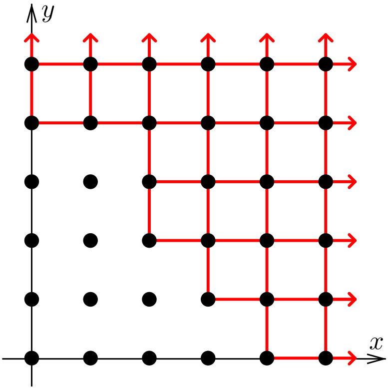

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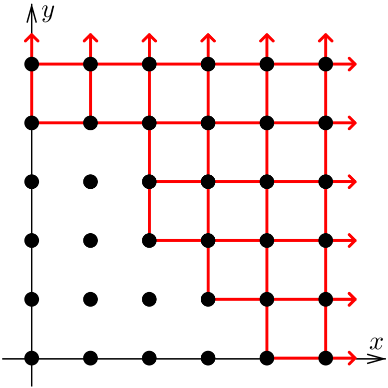

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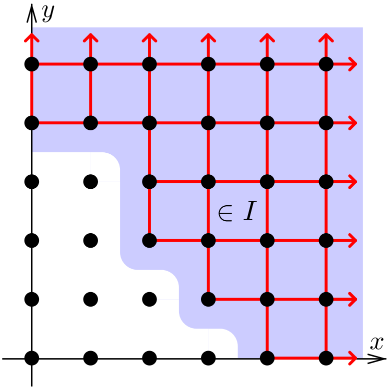

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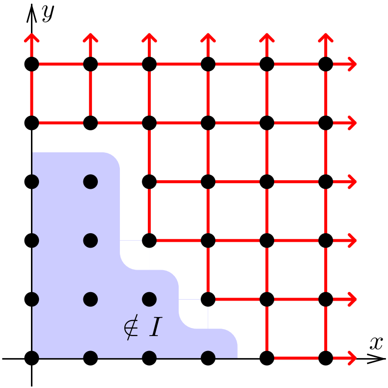

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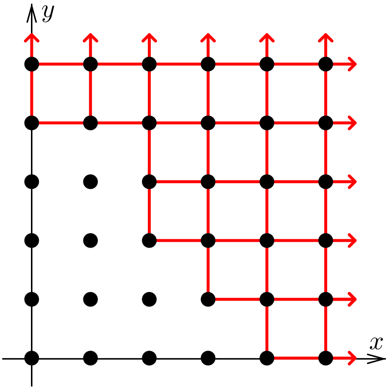

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Generators of I are
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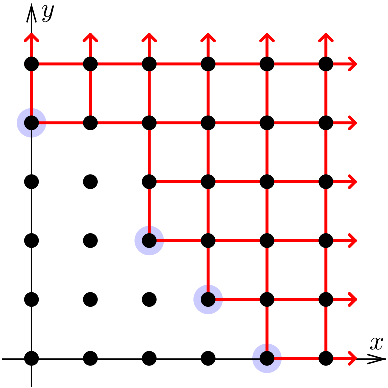

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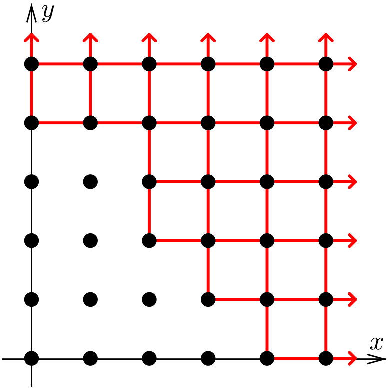

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for irreducible monomial ideals $J_1, \ldots, J_r$.

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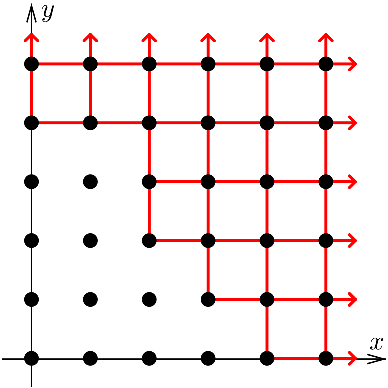

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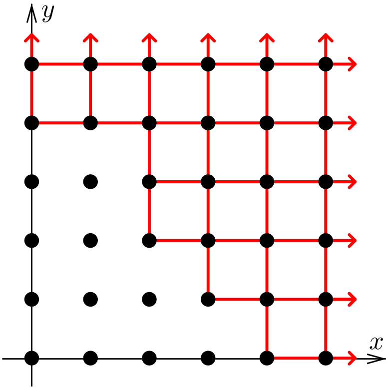

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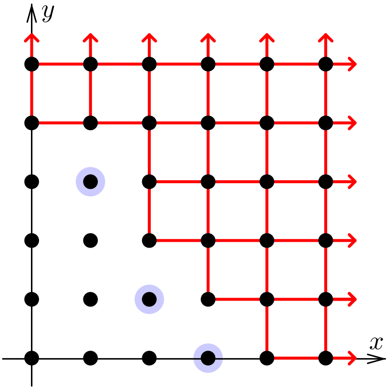

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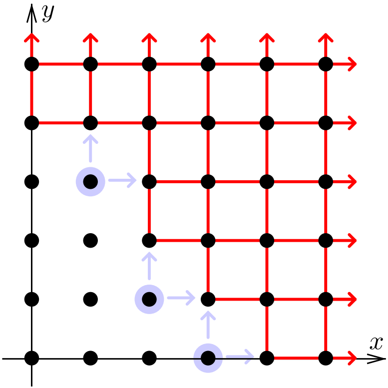

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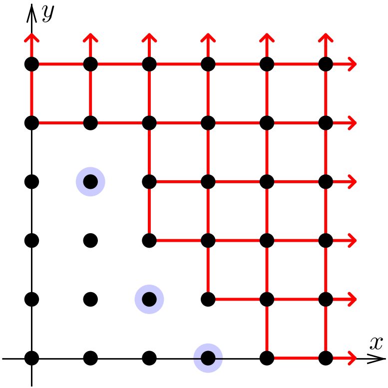

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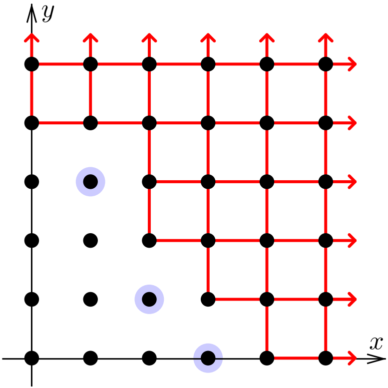

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Irreducible decomposition:
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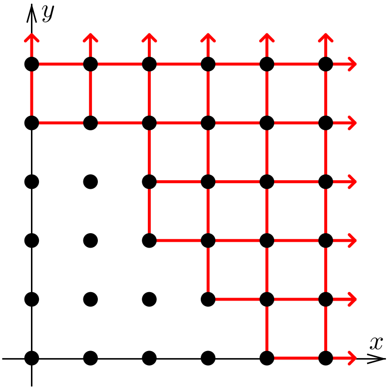

Monomial Ideals

$$I \quad = \quad \langle x^4, x^3y, x^2y^2, y^4 \rangle$$

I is $\mathfrak{p}$-primary, $\mathfrak{p} = \langle x, y \rangle$

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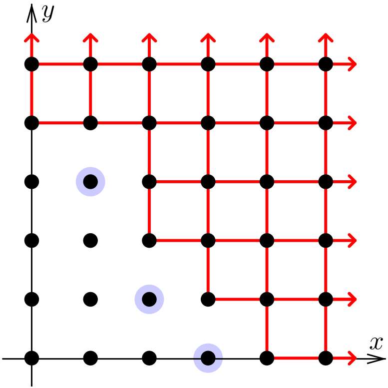

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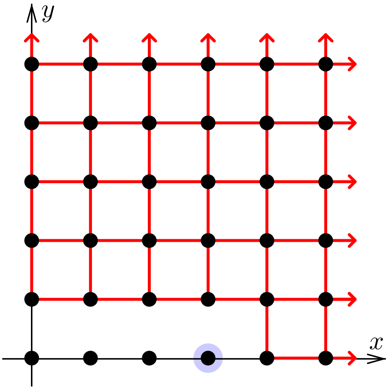

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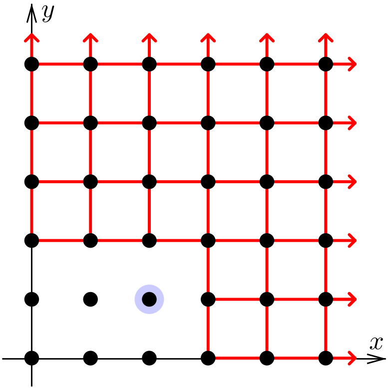

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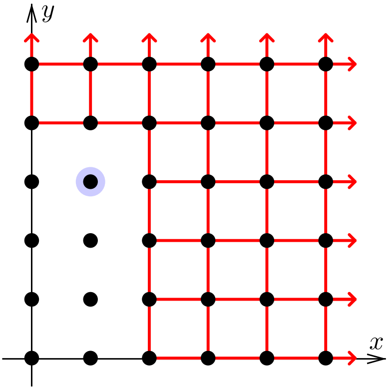

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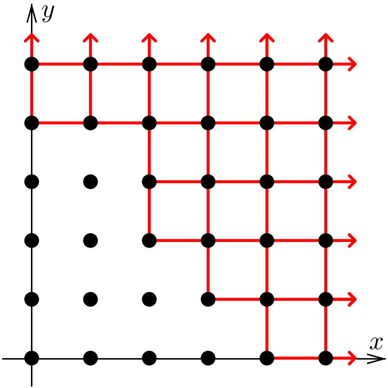

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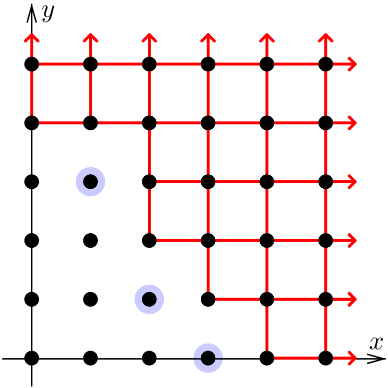

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Binomial ideals

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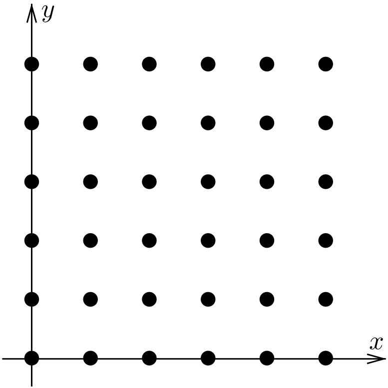

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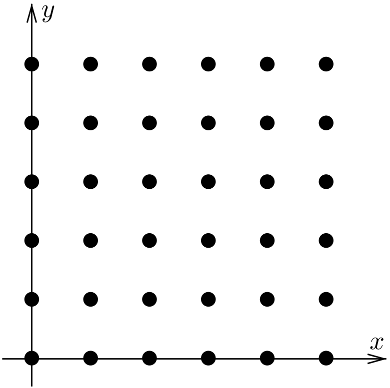

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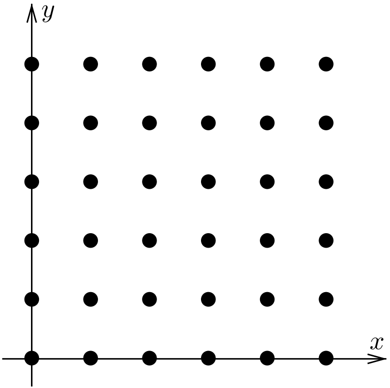

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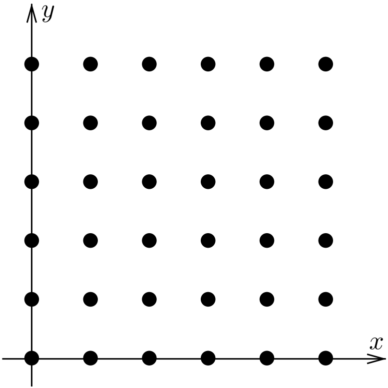

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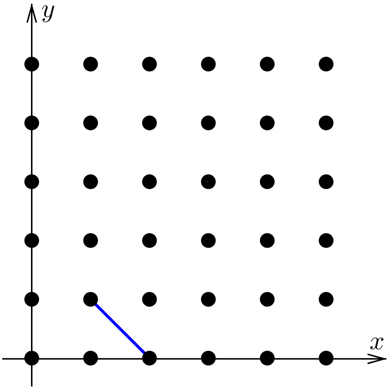

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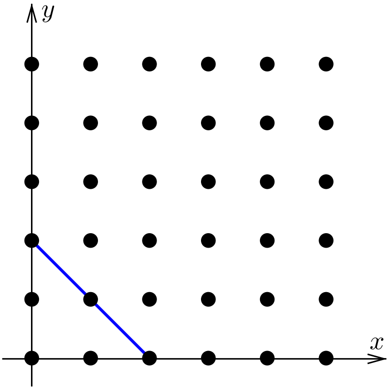

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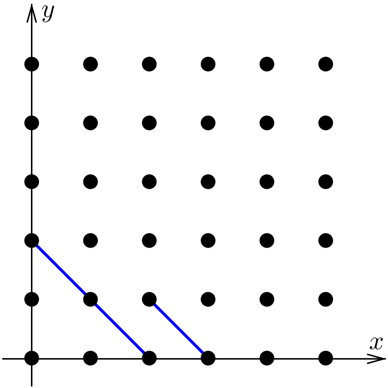

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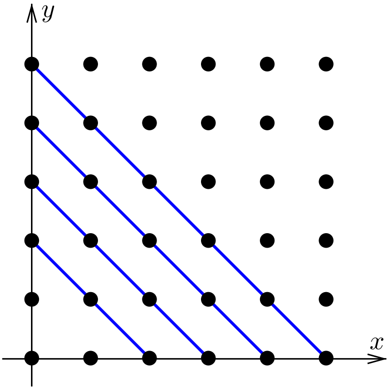

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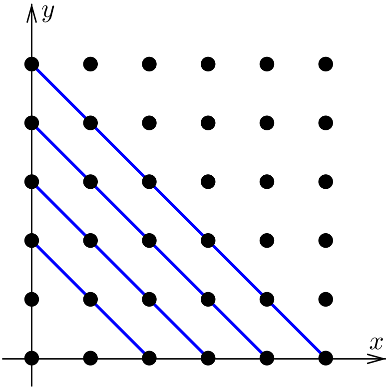

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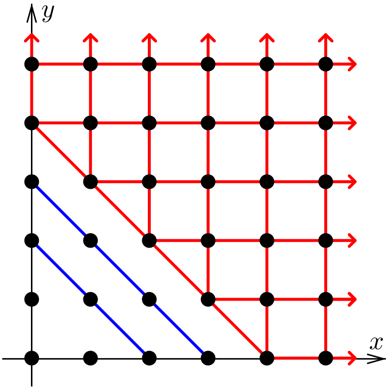

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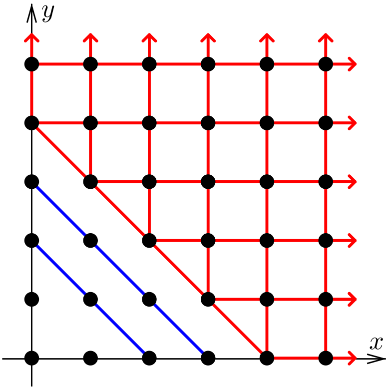

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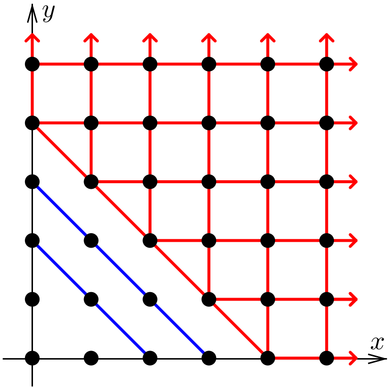

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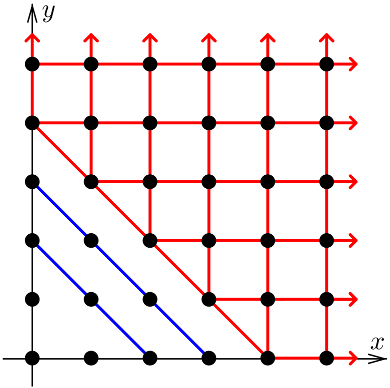
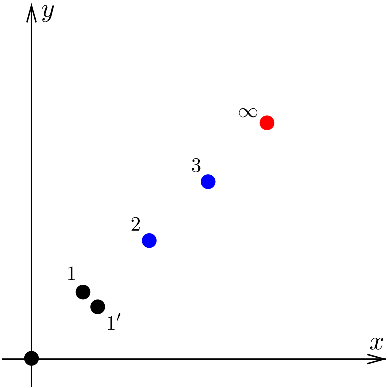
Monoid $\mathbb{N}^2/\sim_I$





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For $\mathbb{k} = \overline{\mathbb{k}}$, every binomial ideal has an expression of the form

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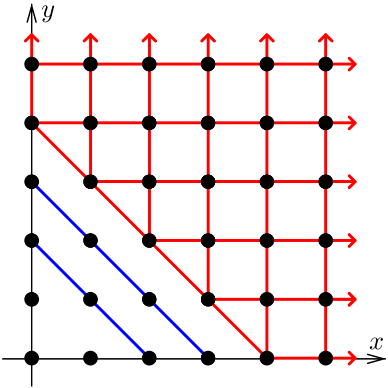

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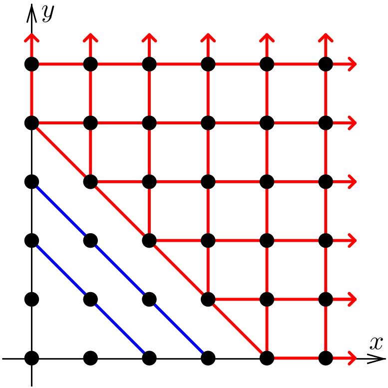

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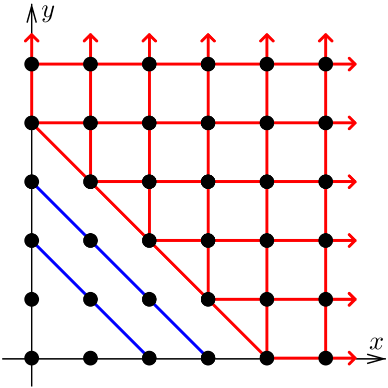

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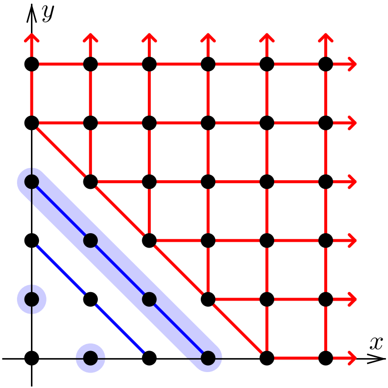

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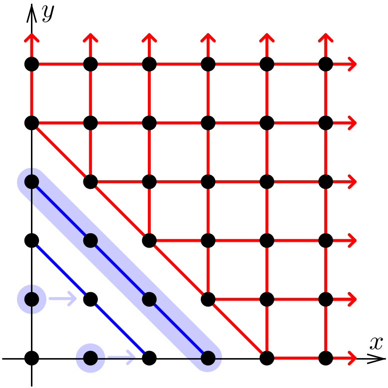

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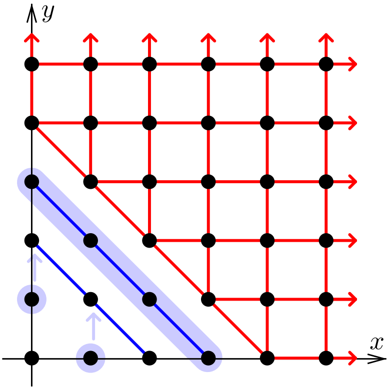

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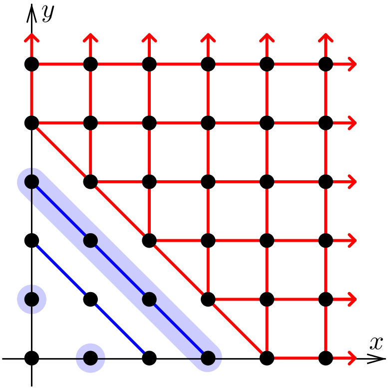

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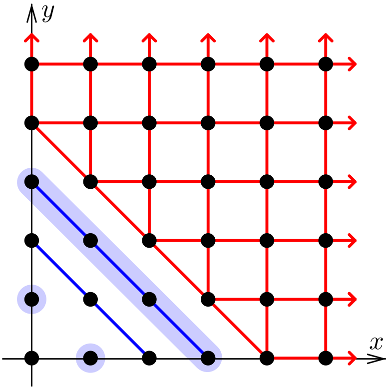

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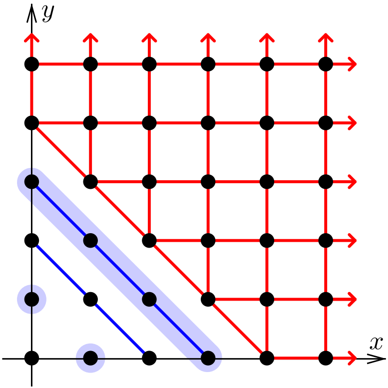
I-witnesses: x^3, x, y



Binomial Ideals

Definition

A monomial $\mathbf{x}^{\mathbf{a}}$ is a *witness* for I if for each $\mathbf{x}^{\mathbf{p}} \in \mathfrak{p}$,

$$\mathbf{p} + \mathbf{a} \sim_I \mathbf{p} + \mathbf{a}' \text{ for some } \mathbf{a}' \not\sim_I \mathbf{a},$$

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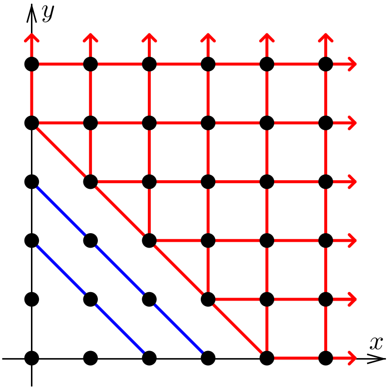
Theorem (Kahle-Miller, 2013)

For any $\mathfrak{p}$-primary binomial ideal I, any $f \in \operatorname{soc}_\mathfrak{p}(I)/I$ is a sum of witnesses.

Binomial Ideals

$$I = \langle x^2 - xy, xy - y^2, x^4, y^4 \rangle$$



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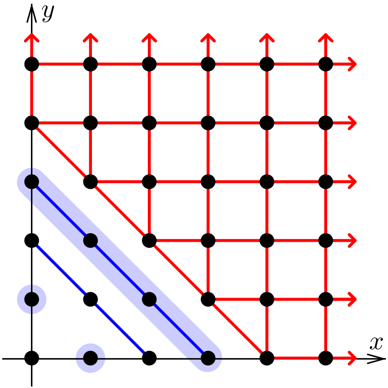

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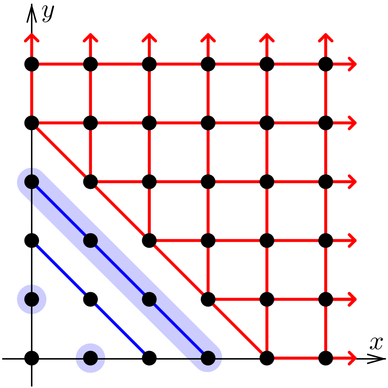
soccularize I: "Force simple socle"



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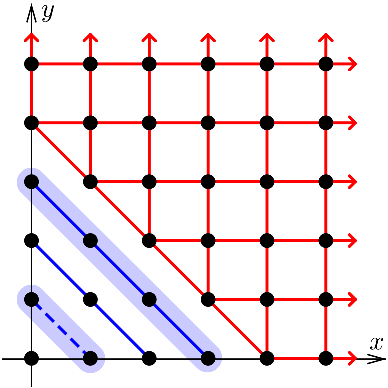

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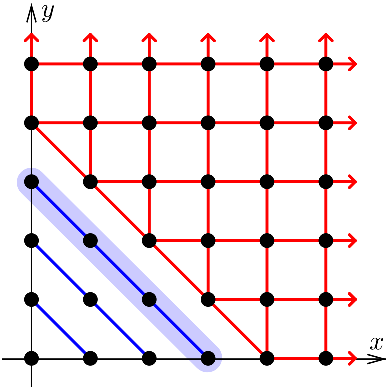

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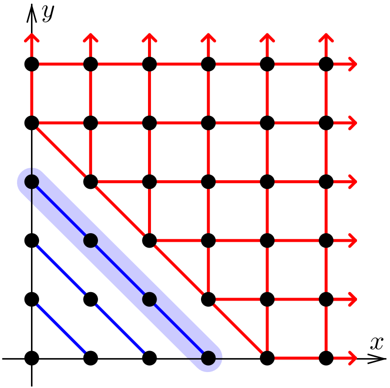

Soccular Decomposition

Plan of attack:

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- One irreducible component per witness monomial.
- For each component, force chosen witness to be maximal.
- Soccularize to remove other socle elements.

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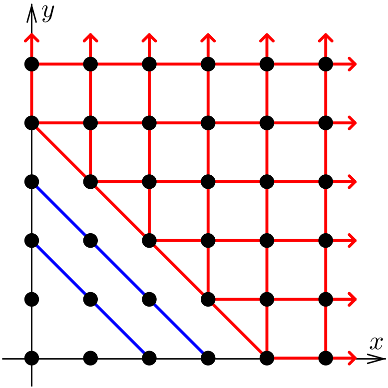

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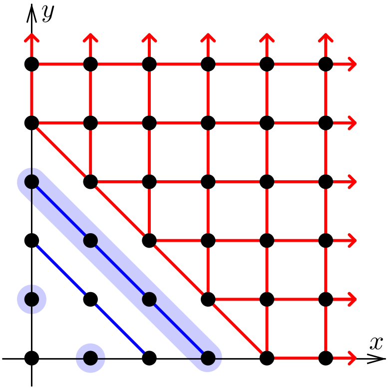

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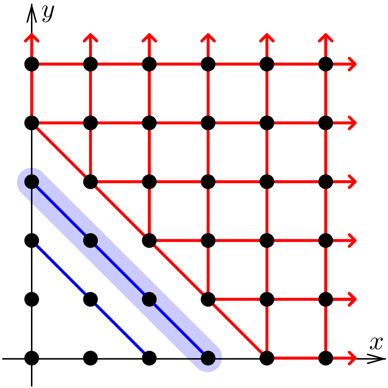

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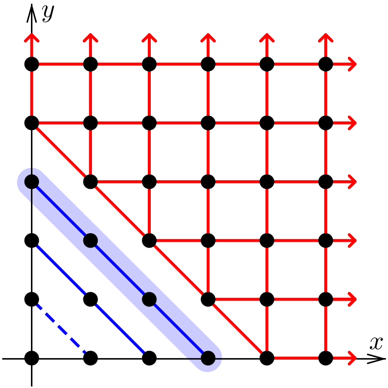

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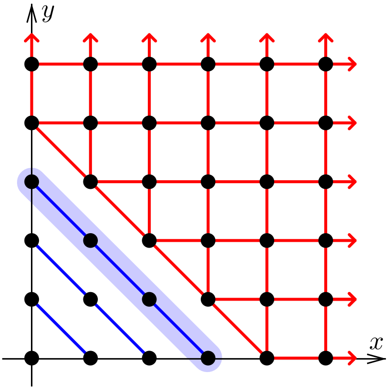

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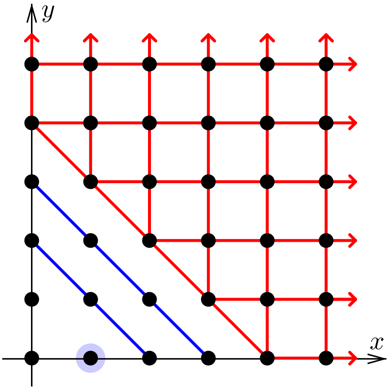

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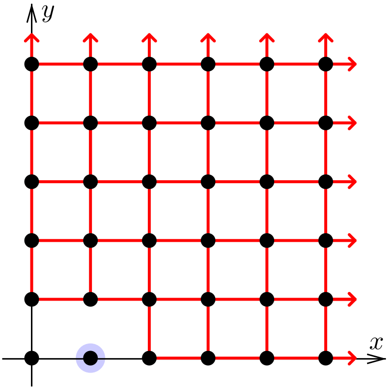

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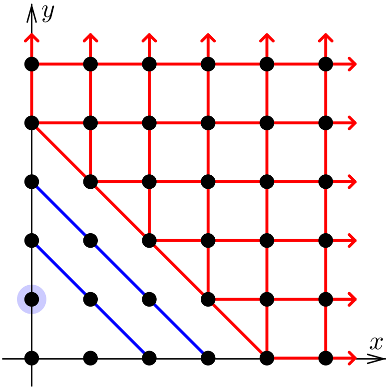

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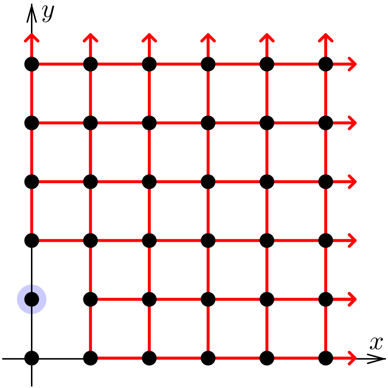

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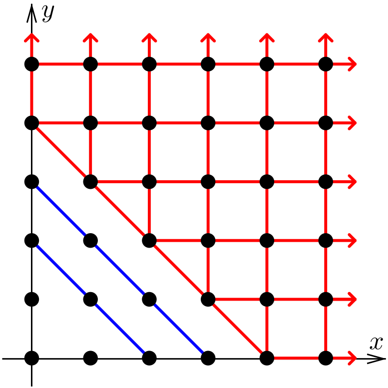
$$I \quad = \quad J_1 \cap J_2 \cap J_3$$



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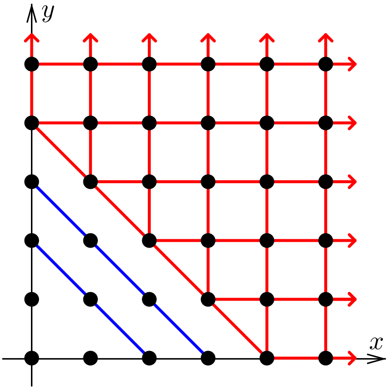
$$\begin{aligned} I &= J_1 \cap J_2 \cap J_3 \\ &= J_1 \cap J_2 \end{aligned}$$



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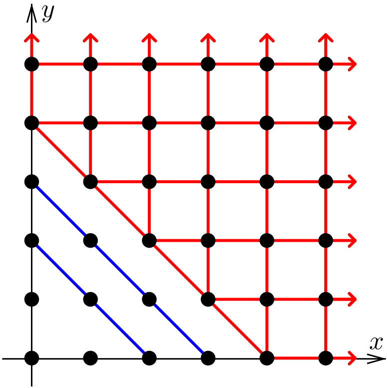

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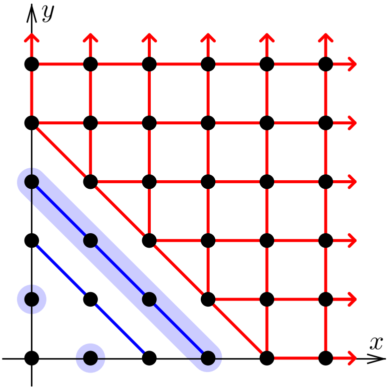

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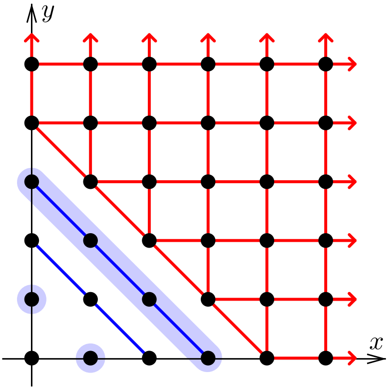

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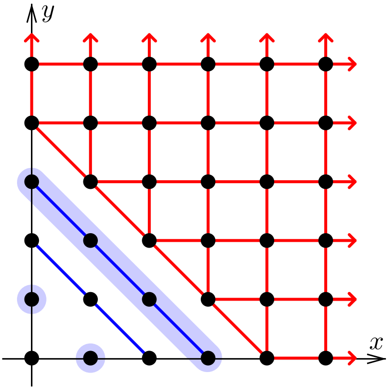

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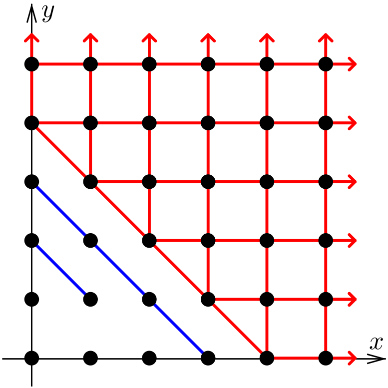

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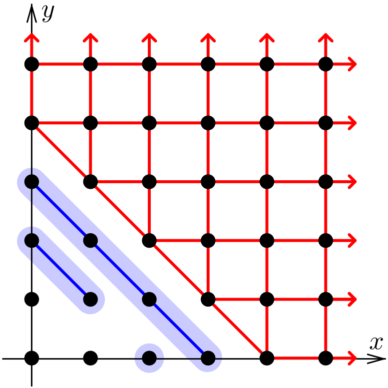

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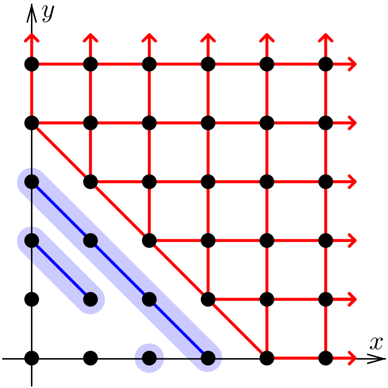

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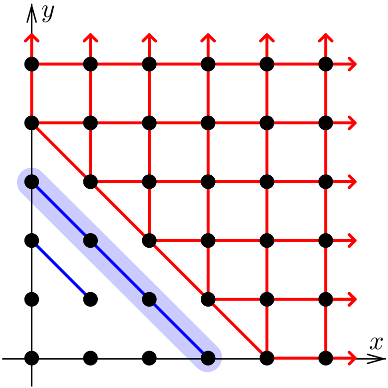

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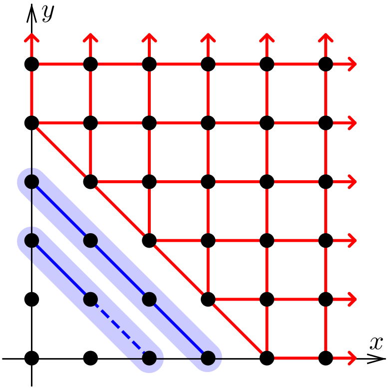

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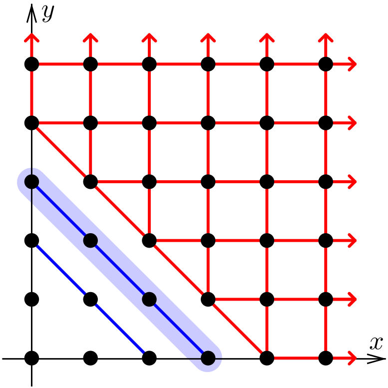

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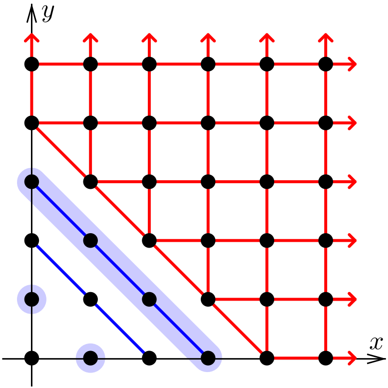
Soccularize: New witnesses!



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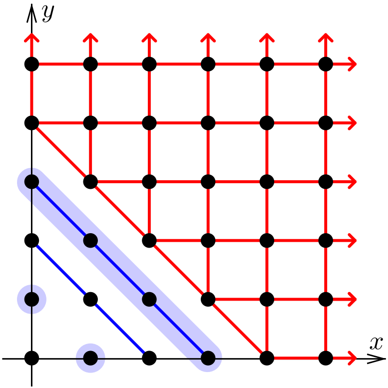

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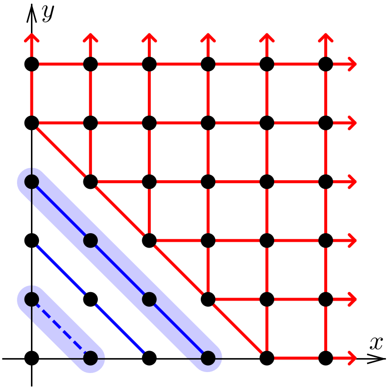

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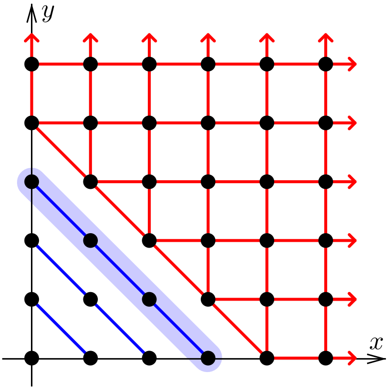

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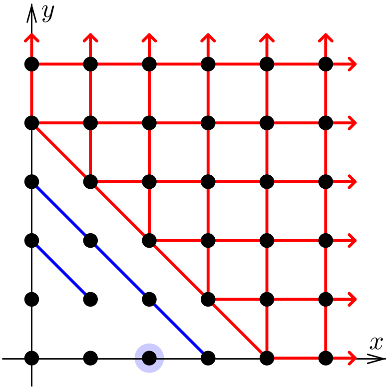

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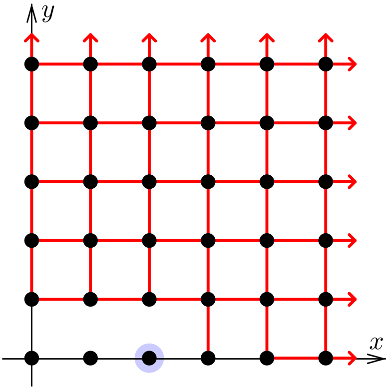

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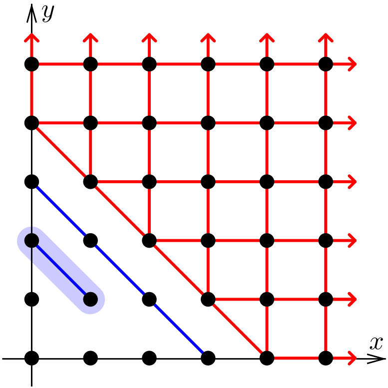

Soccular Decomposition

$$I \quad = \quad \langle xy - y^2, x^3 - xy^2, x^4, y^4 \rangle$$

Witnesses: x^3, x^2, $\mathbf{xy}$

$$\mathrm{soc}_{\mathfrak{m}}(I)/I = \Bbbk\{x^3, x^2 - xy\}$$

Soccularize: New witnesses!
Protected witnesses: x, y

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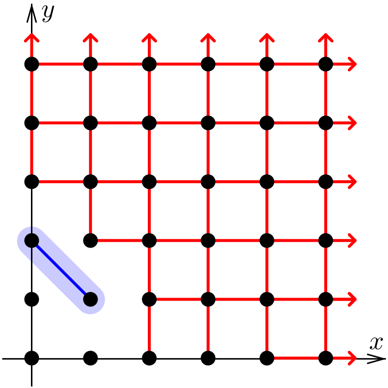

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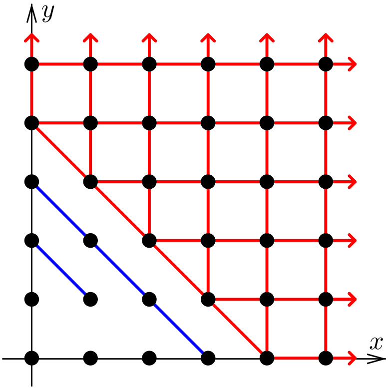

Algorithm for decompositing a binomial ideal I:

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Theorem (Kahle-Miller-O., 2014)

For $\mathbb{k} = \overline{\mathbb{k}}$, any binomial ideal I can be written as $I = \bigcap_{i=1}^{r} J_i$, where each J_i is binomial and $\mathfrak{p}_i$-primary, and the socle $\mathrm{soc}_{\mathfrak{p}_i}(J_i)/J_i$ contains a unique monomial and no other binomials.

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$$I = \langle x^2y - xy^2, x^3, y^3 \rangle$$

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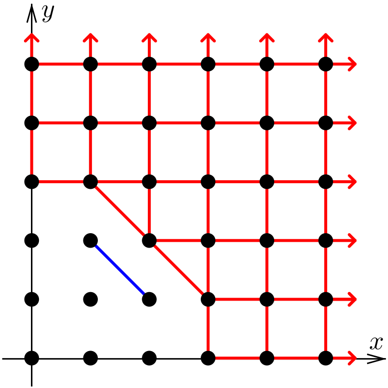

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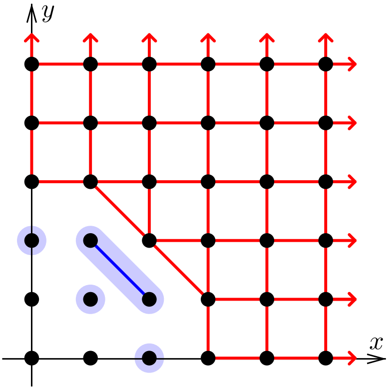

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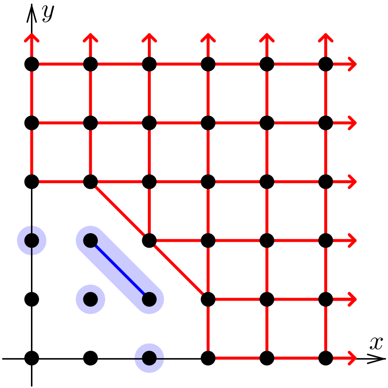

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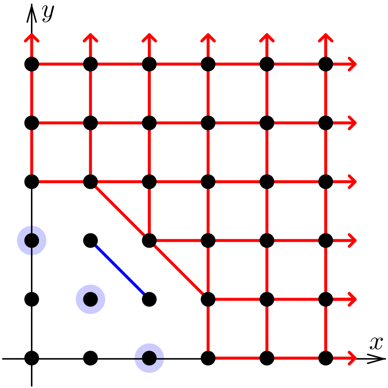

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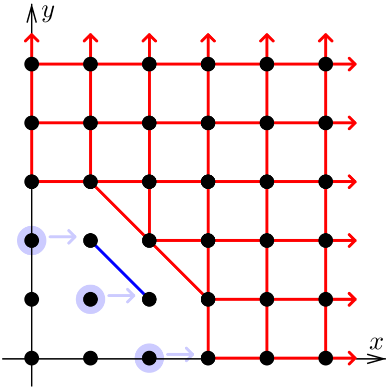

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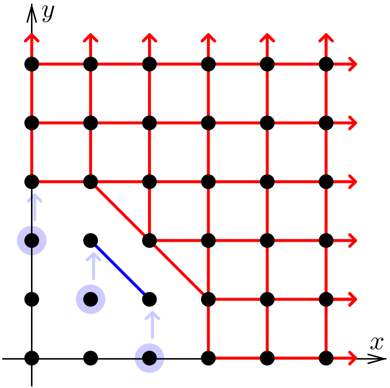

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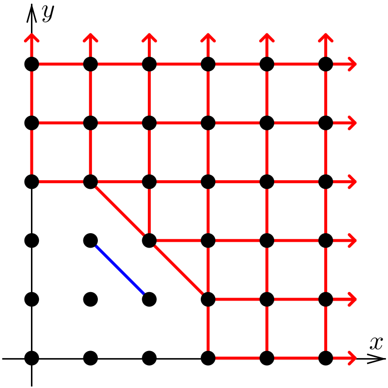

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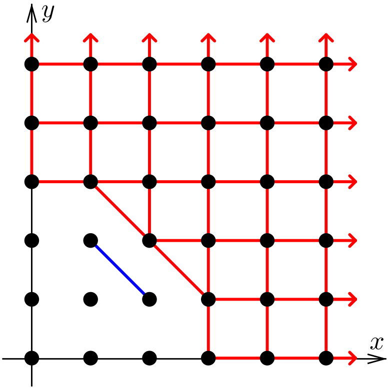

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But $I + \langle \alpha + \lambda\beta \rangle$ already has simple socle, so $J_1 = I + \langle \alpha + \lambda\beta \rangle$. $\square$

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Ezra Miller, Bernd Sturmfels (2005)
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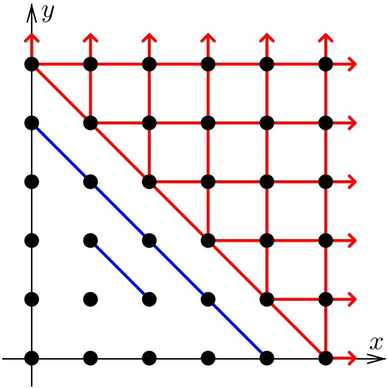

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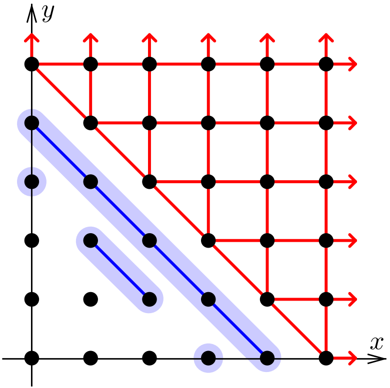

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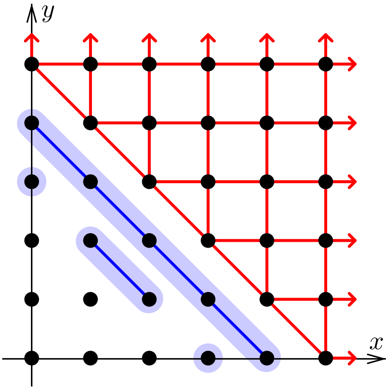

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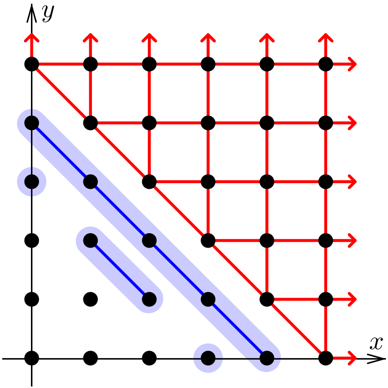
J_3 not binomial



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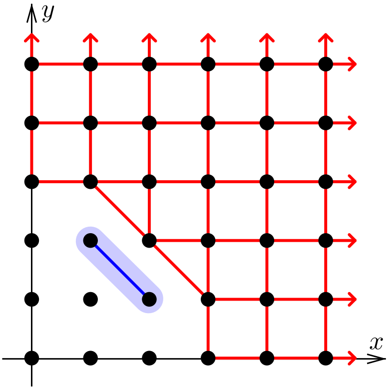

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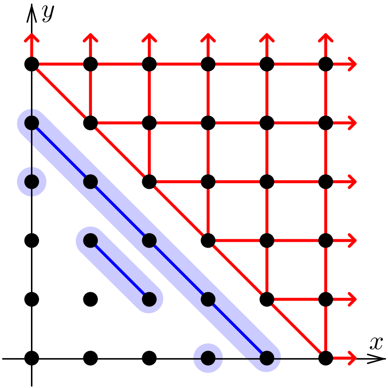

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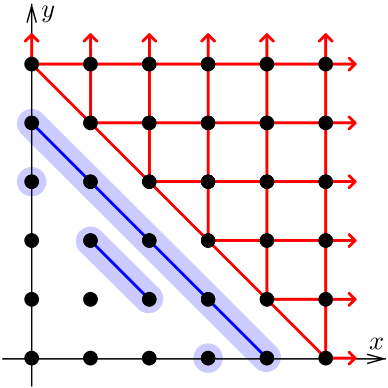

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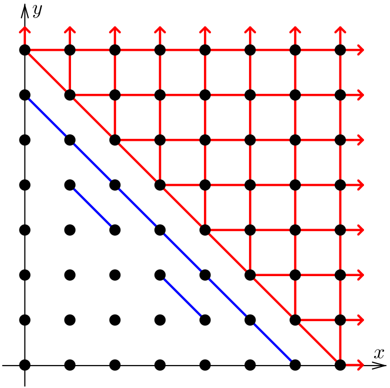

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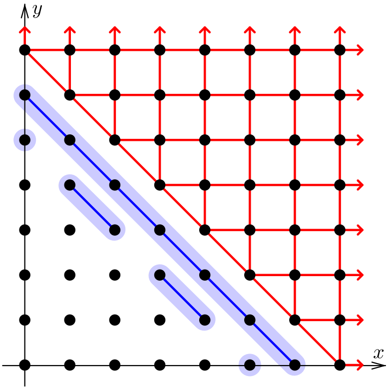

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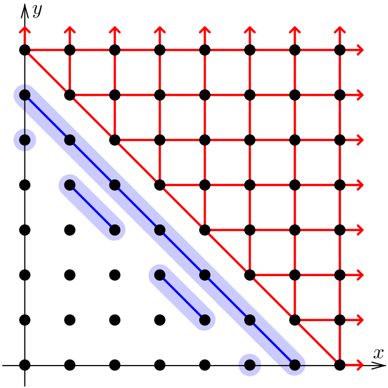

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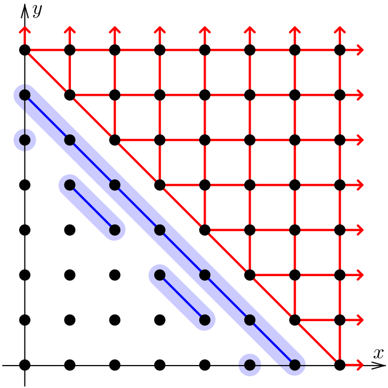

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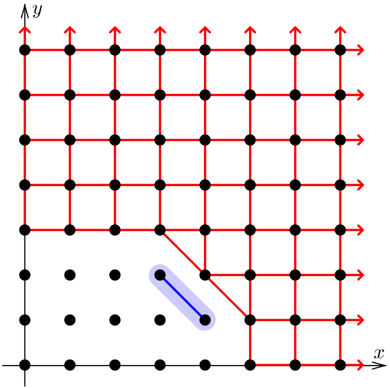

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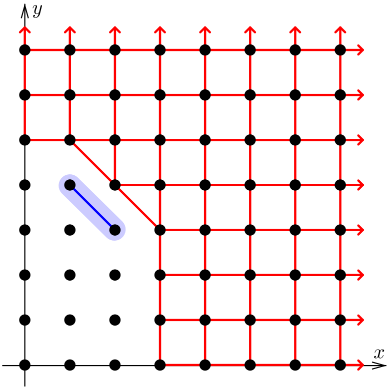

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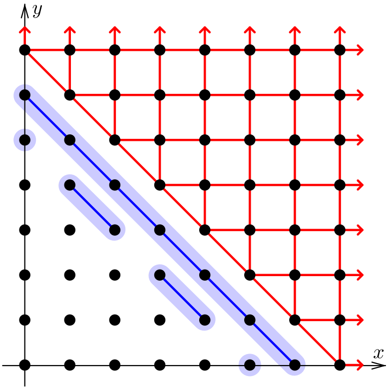

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