Invariants of non-unique factorization

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Joint with Roberto Pelayo

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Factorial domains

Definition

An integral domain R is *factorial* if for each non-unit $r \in R$,

1. there is a *factorization* $r = u_1 \cdots u_k$ as a product of irreducibles, and
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The point: it's nontrivial.

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2. $x^6 = x^3 \cdot x^3 = x^2 \cdot x^2 \cdot x^2$.

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Interesting monoids

Definition

An *arithmetical congruence monoid* is a multiplicative submonoid

$$M_{a,b} = \{n : n \equiv a \bmod b\} \subset \mathbb{Z}_{>0}$$

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$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$.

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$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$. "McNugget Monoid"

Factorization invariants

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Fix a commutative, cancellative monoid $(M, \cdot)$. For non-unit $m \in M$,

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- This is (almost) the best we could hope for.

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- $\rho(n) \le n_k/n_1$ for all $n \in S$.

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- $\rho(m) < 2$ for all $m \in M_{4,6}$!
- Elasticity of $M_{4,6}$ is *not accepted*.

Factorization invariants

Definition (ω-primality)

Fix a cancellative, commutative, atomic monoid M. For $x \in M$, $\omega(x)$ is the smallest positive integer m such that whenever $x \mid \prod_{i=1}^{r} u_i$ for $r > m$, there exists a subset $T \subset \{1, \ldots, r\}$ with $|T| \leq m$ such that $x \mid \prod_{i \in T} u_i$.

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$\omega(x) = 1$ if and only if x is prime (i.e. $x \mid ab$ implies $x \mid a$ or $x \mid b$).

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Fact

M is factorial if and only if every irreducible element of M is prime. Moreover, $\omega(p_1 \cdots p_r) = r$ for any primes $p_1, \ldots, p_r \in M$.

Quasilinearity for numerical monoids

Theorem (O.–Pelayo, 2013)

Fix a numerical monoid $S = \langle n_1, \ldots, n_k \rangle \subset \mathbb{N}$. *For* $n \gg 0$,

$$\omega_S(n) = \frac{1}{n_1} n + a_0(n)$$

where $a_0(n)$ *periodic with period* n_1.

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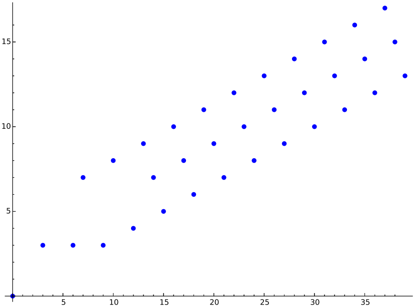
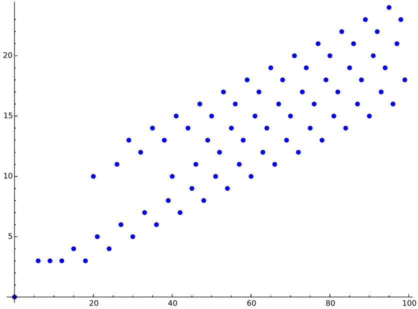


$S = \langle 3, 7 \rangle$



$McN = \langle 6, 9, 20 \rangle$

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Christopher O'Neill, Roberto Pelayo (2014)
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Manuel Delgado, Pedro García-Sánchez, Jose Morais
GAP Numerical Semigroups Package
`http://www.gap-system.org/Packages/numericalsgps.html`.

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