Catenary degrees of elements in numerical monoids

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Non-unique factorization

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Fix a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$,

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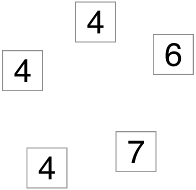

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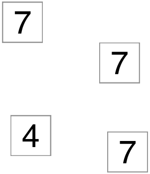

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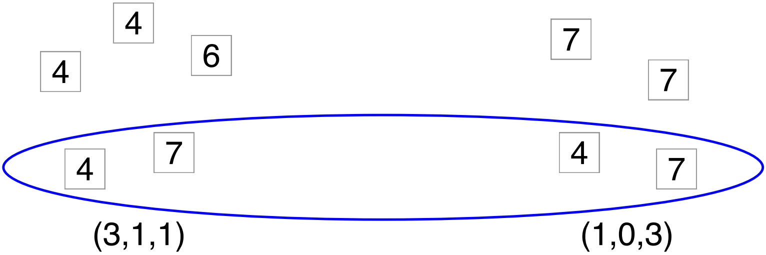

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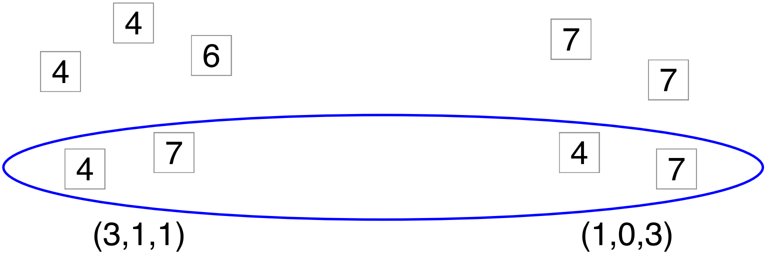

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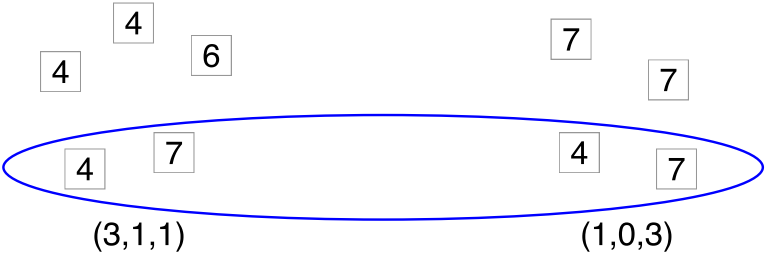

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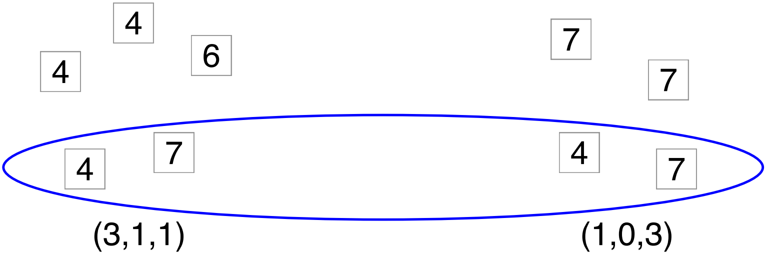

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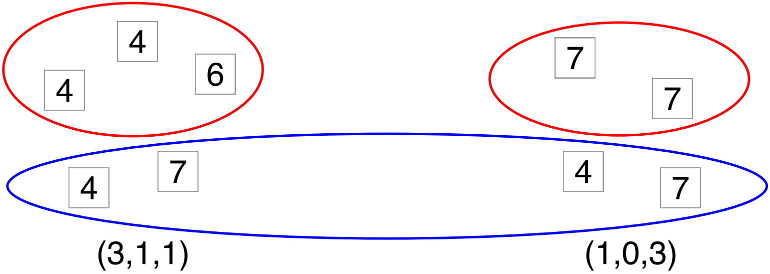

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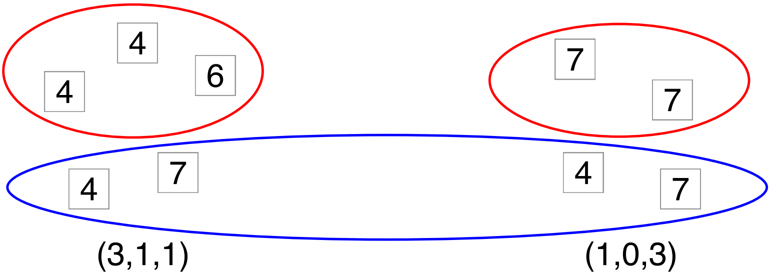

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If $|\mathsf{Z}_S(n)| = 1$, define $c(n) = 0$.

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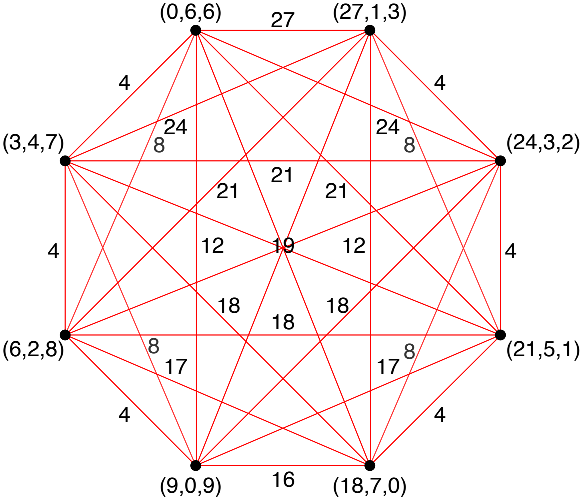

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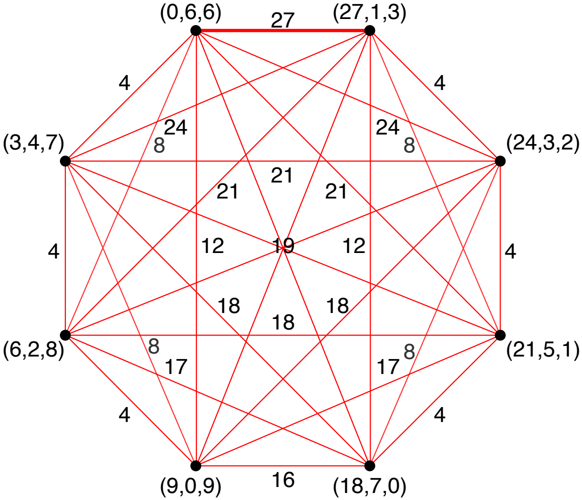

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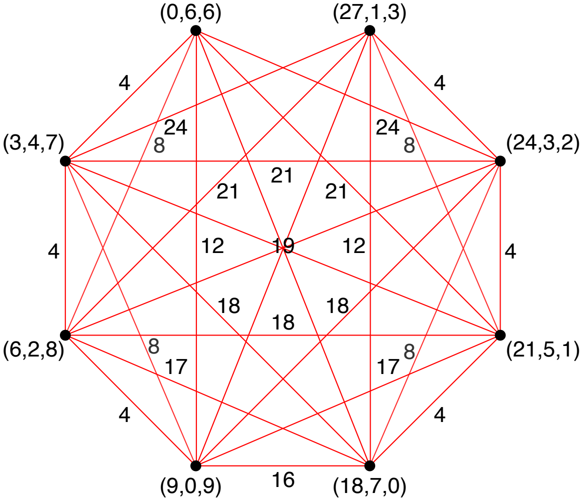

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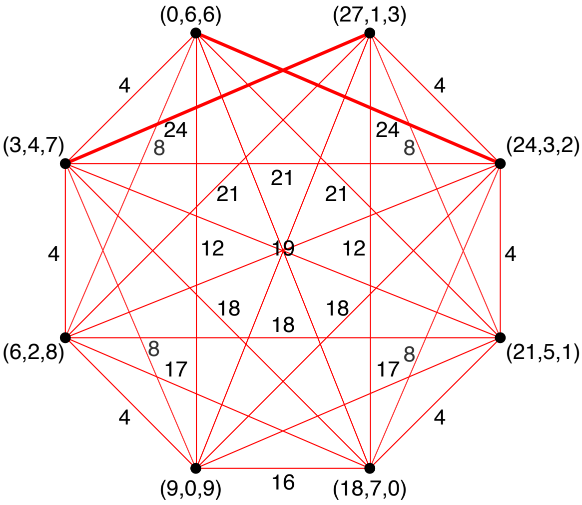

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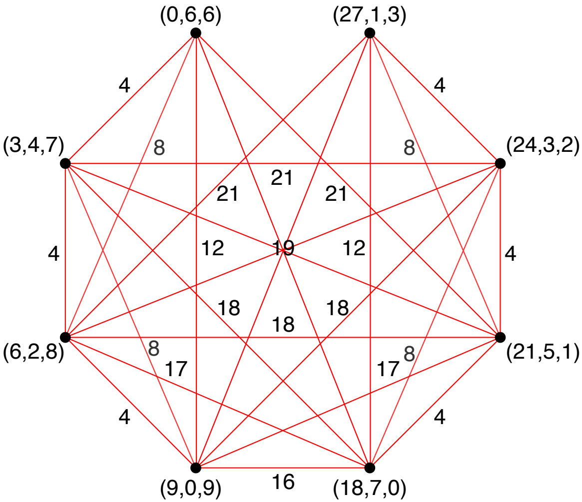

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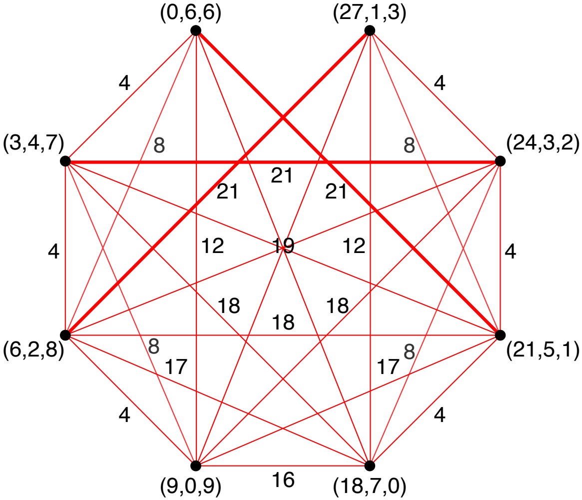

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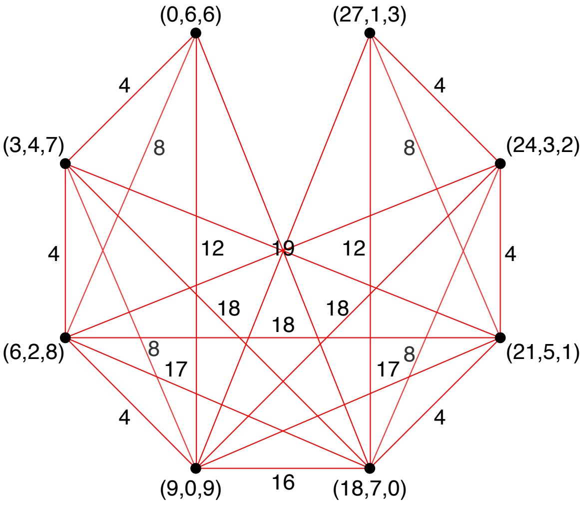

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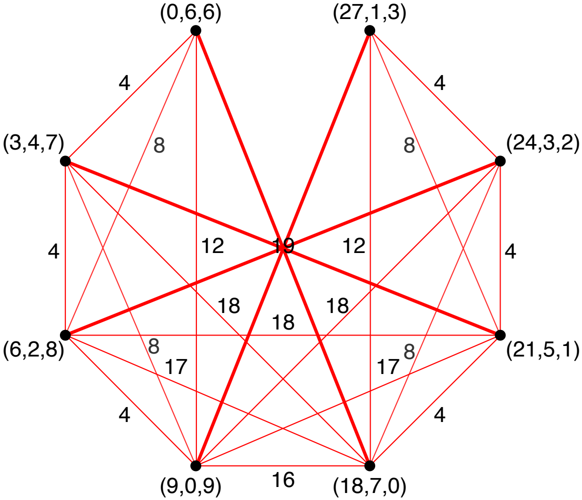

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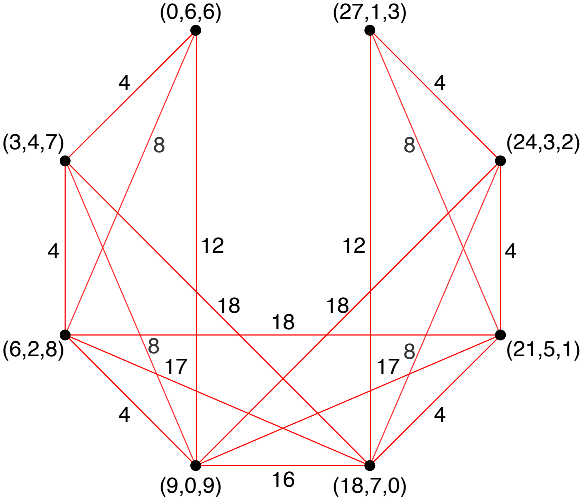

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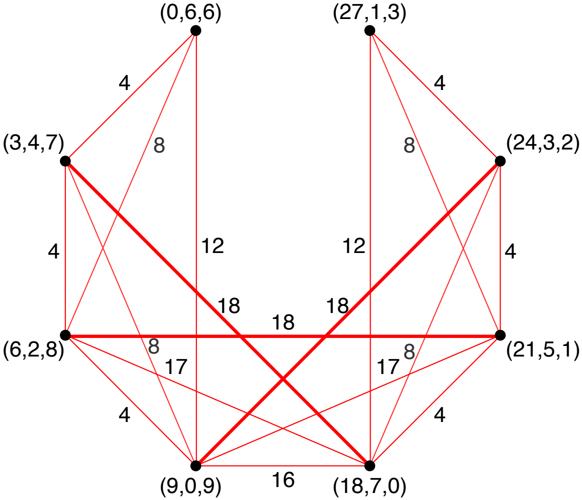

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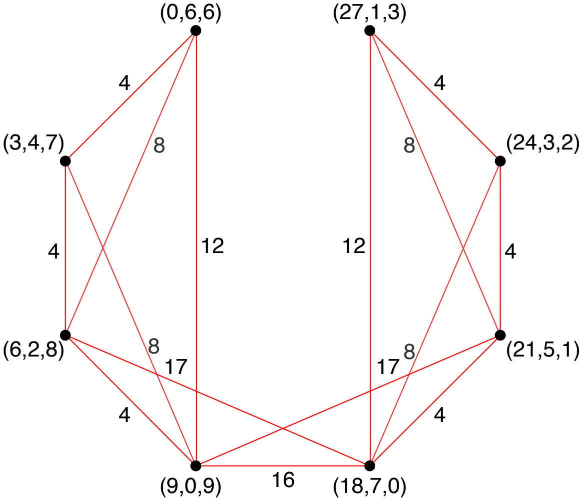

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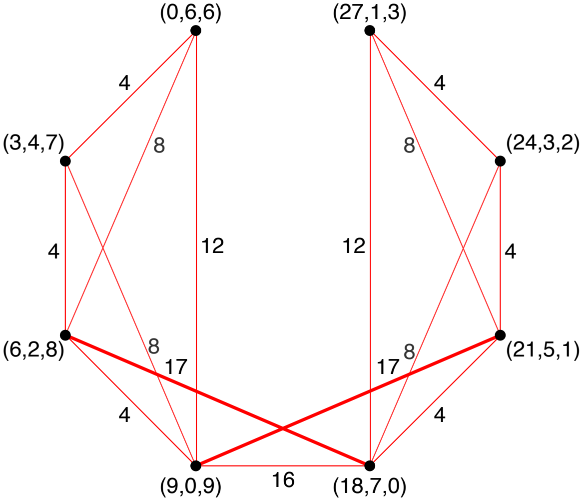

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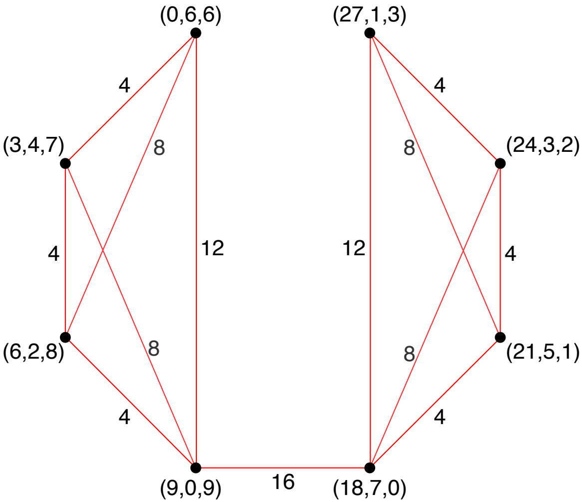

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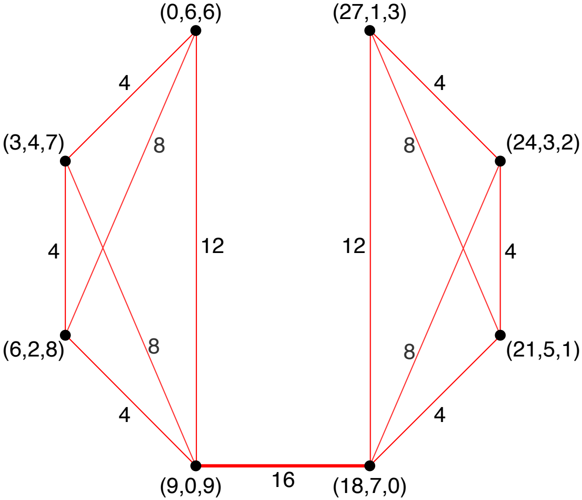

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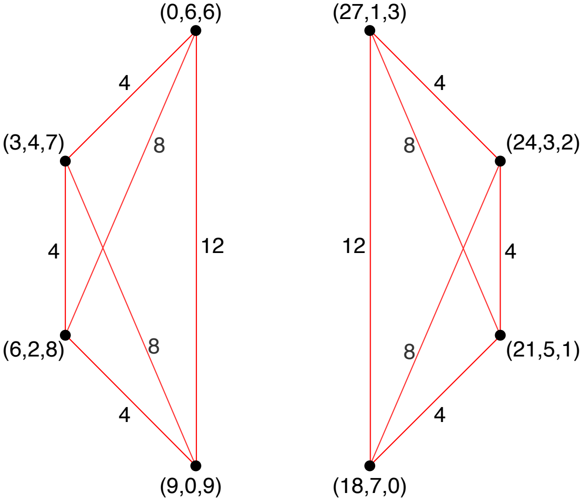

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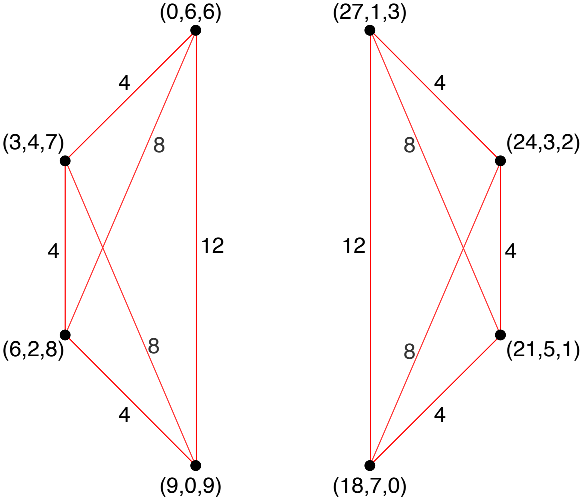

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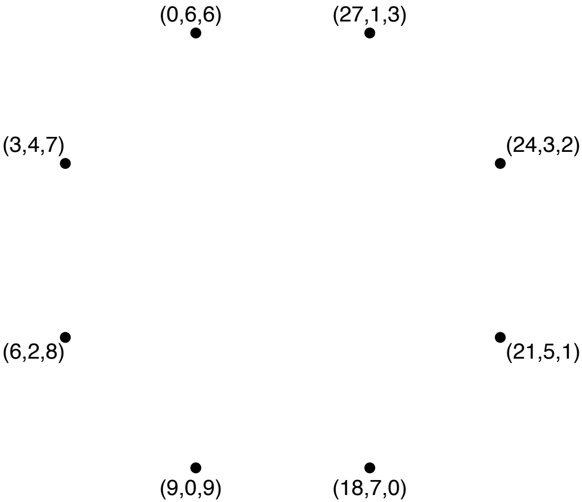

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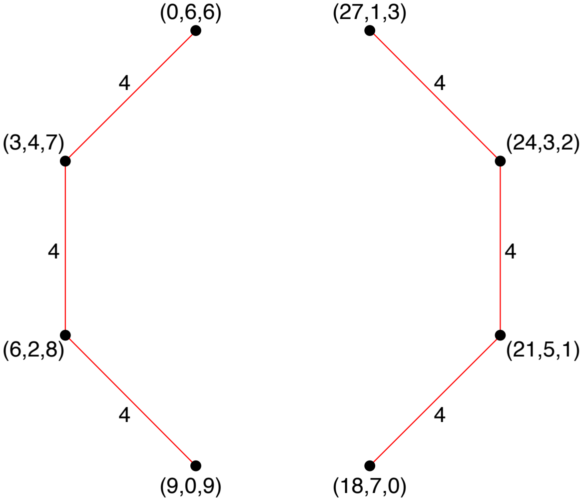

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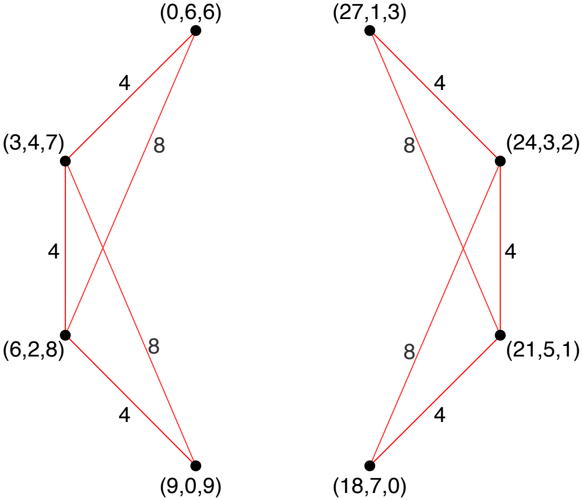

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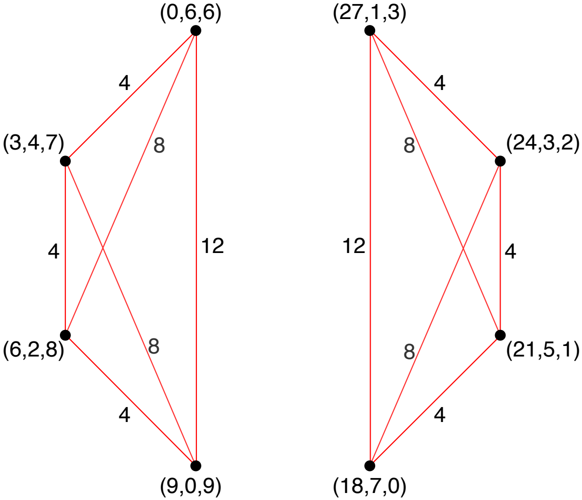

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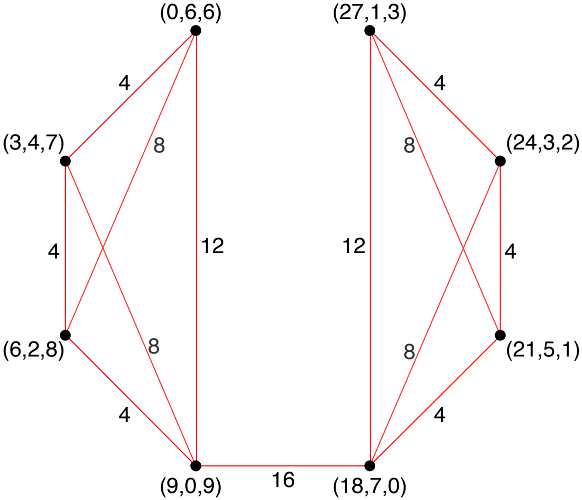

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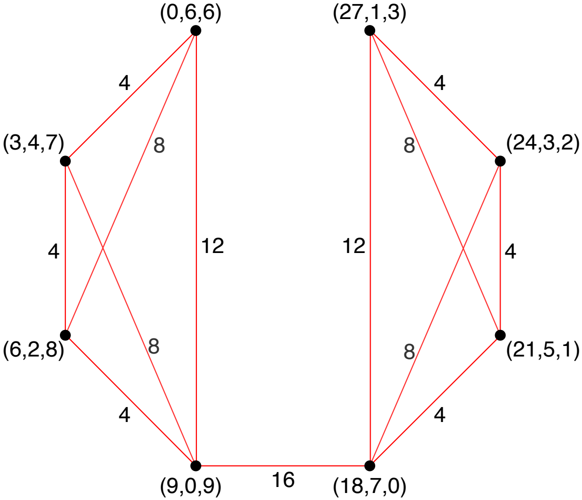

Betti elements

Definition

For an element $n \in S = \langle n_1, \ldots, n_k \rangle$, let ∇_n denote the unlabeled graph with vertex set $\mathsf{Z}_S(n)$ and an edge (f, f') whenever $\gcd(f, f') \neq 0$.

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$S = \langle 10, 15, 17 \rangle$ has Betti elements 30 and 85.

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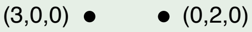

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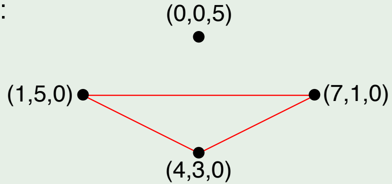

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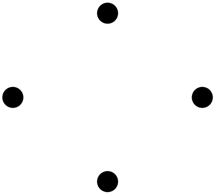
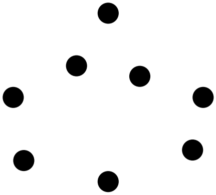


$Z_S(n)$



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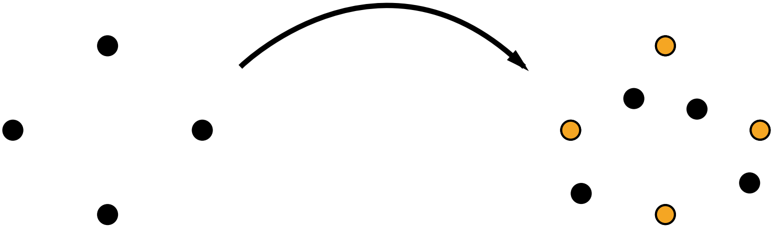

$$Z_S(n) \hookrightarrow Z_S(n + n_i)$$

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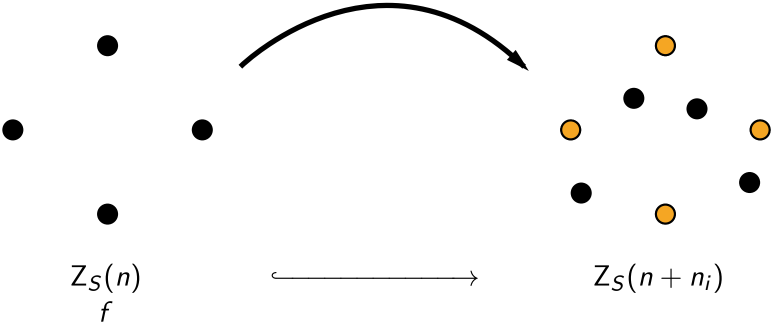

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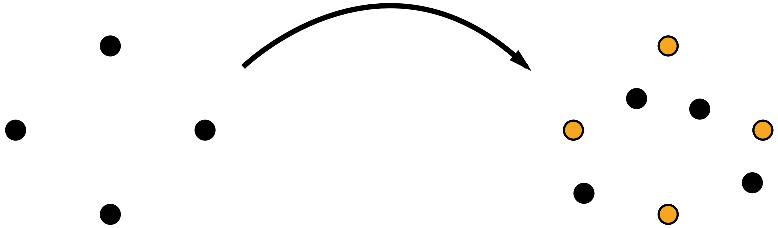

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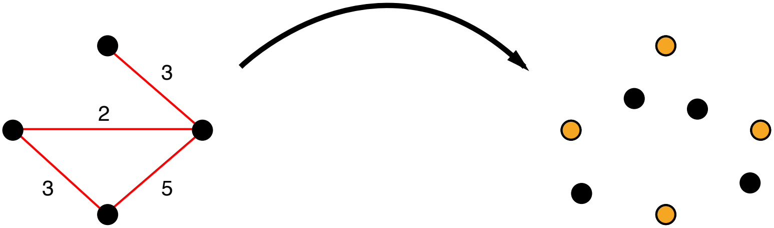

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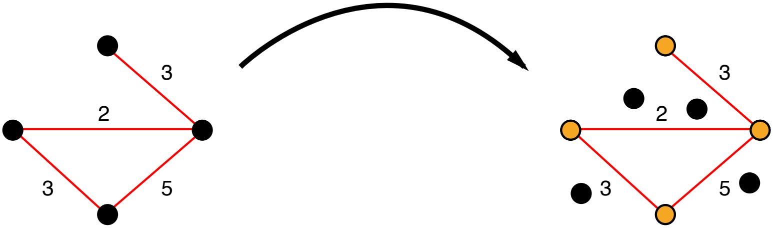

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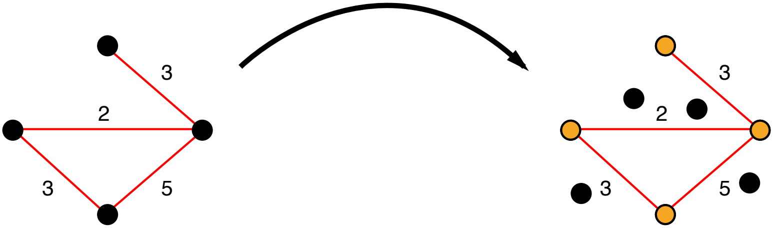

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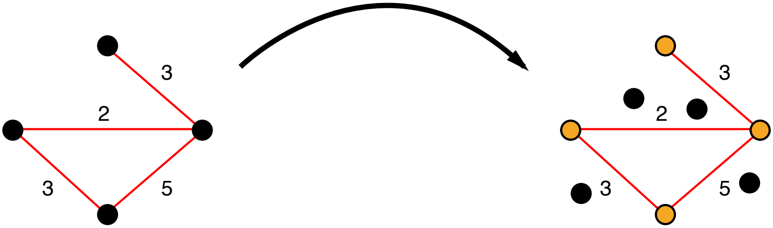

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Idea for proof of Theorem: Images of edges between connected components in ∇_b "span" the catenary graph of each $n \in S$.

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Conjecture

$\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}$.

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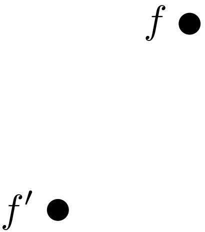

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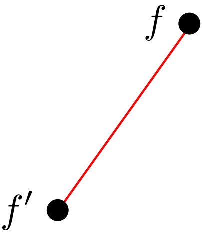
If $f, f' \in \mathsf{Z}_S(n)$ and $d(f, f') < B$,



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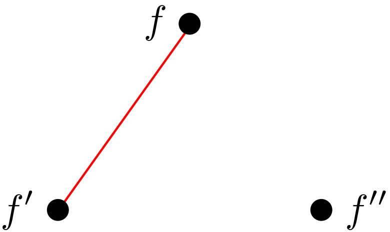

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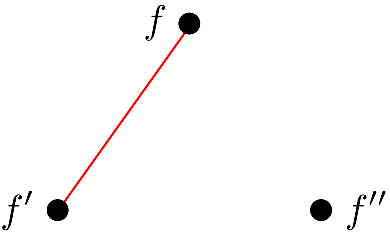

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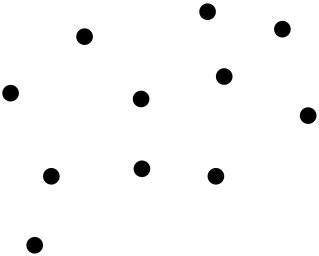

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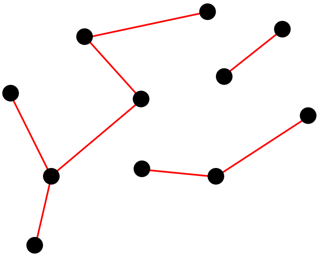

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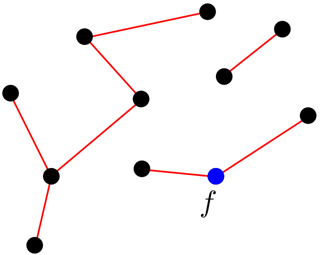

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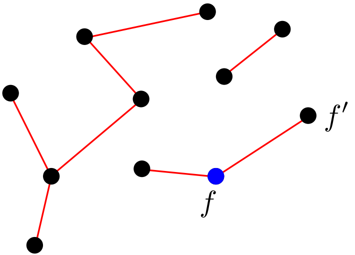

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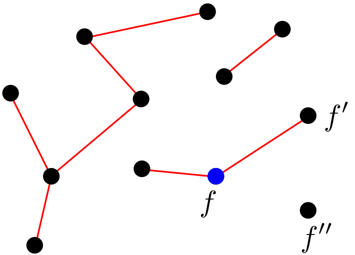

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- maximality of $|f| \Rightarrow$
  f'' has no edges!

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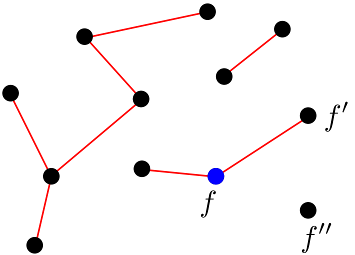

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References

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Scott Champan, Pedro García-Sánchez, David Llena, Vadim Ponomarenko, José Rosales (2006)
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