Invariants of non-unique factorization

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First half: Joint with Vadim Ponomarenko, Reuben Tate*, and Gautam Webb*

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Second half: Joint with Thomas Barron* and Roberto Pelayo

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Factorial domains

Definition

An integral domain R is *factorial* if for each non-unit $r \in R$,

1. there is a *factorization* $r = u_1 \cdots u_k$ as a product of irreducibles, and
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The point: it's complicated.

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Factorization invariants: towards the catenary degree

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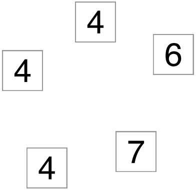

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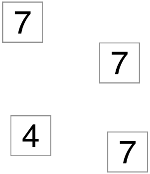

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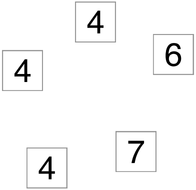

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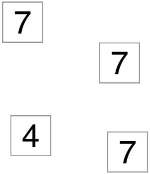

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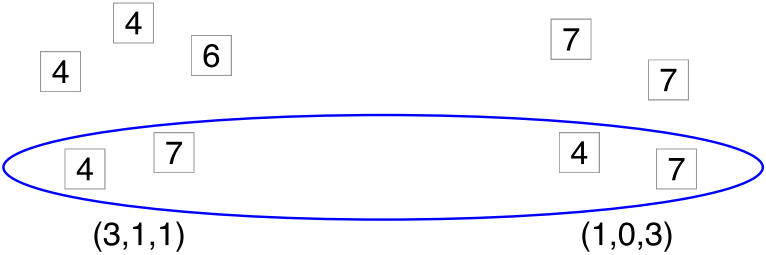

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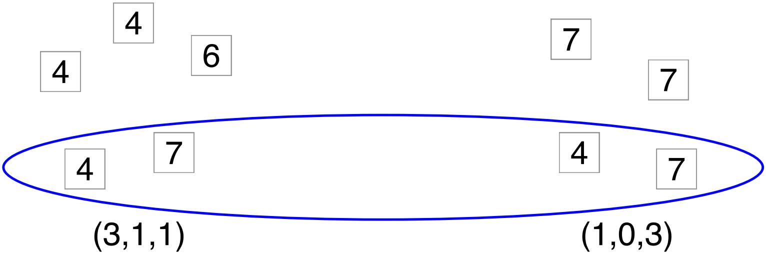

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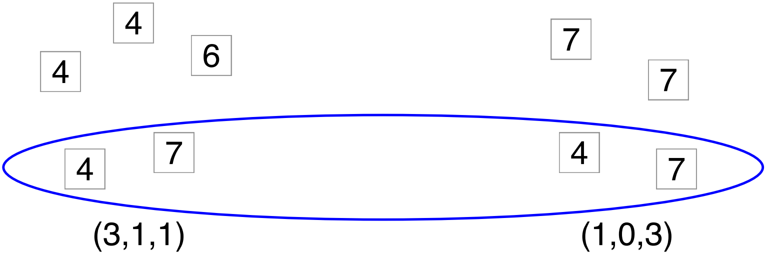

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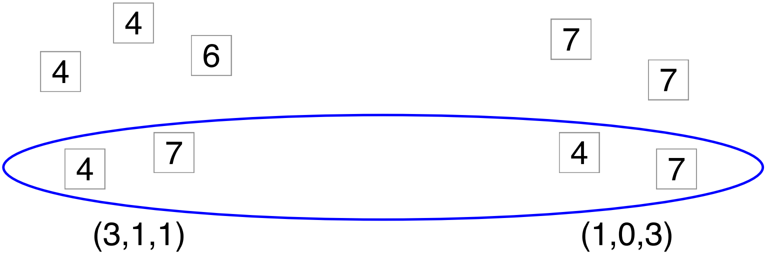

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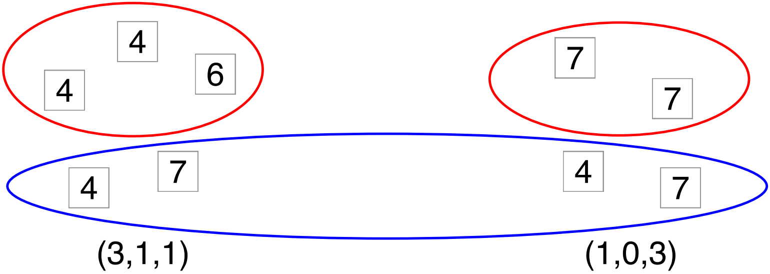

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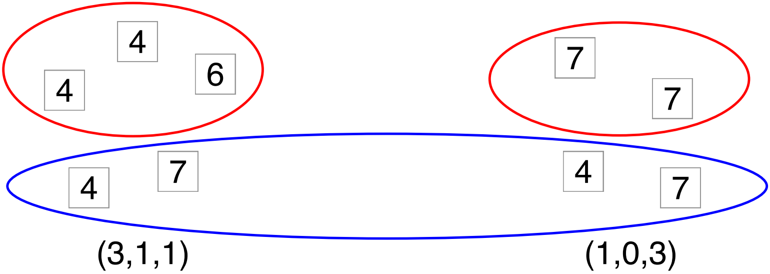

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If $|\mathsf{Z}_S(n)| = 1$, define $c(n) = 0$.

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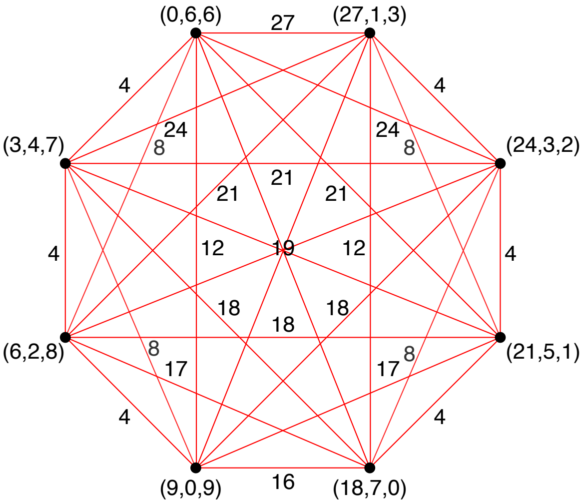

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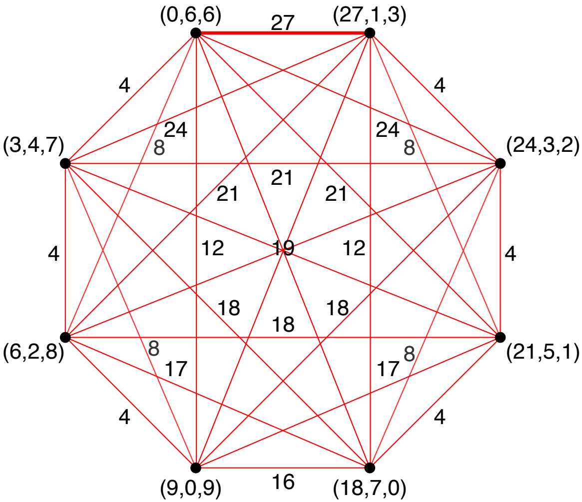

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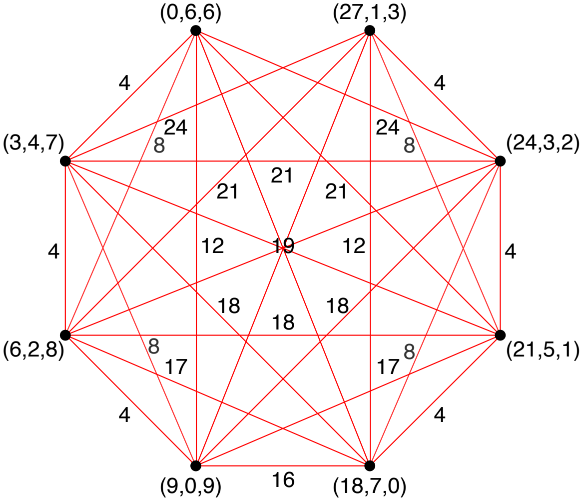
$S = \langle 11, 36, 39 \rangle, n = 450$



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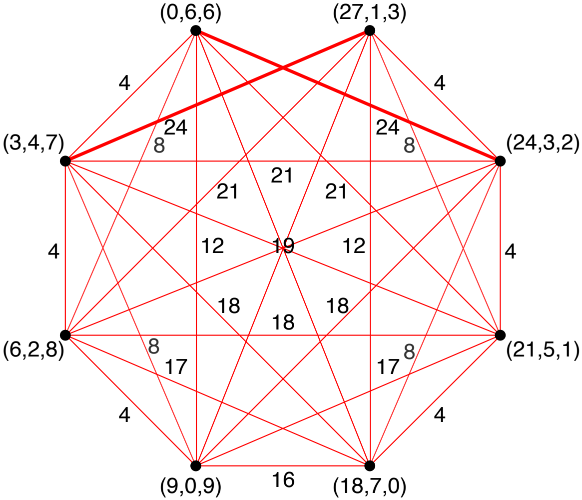

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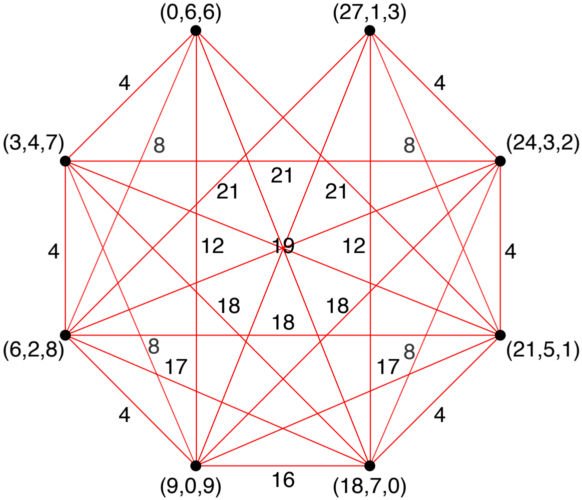

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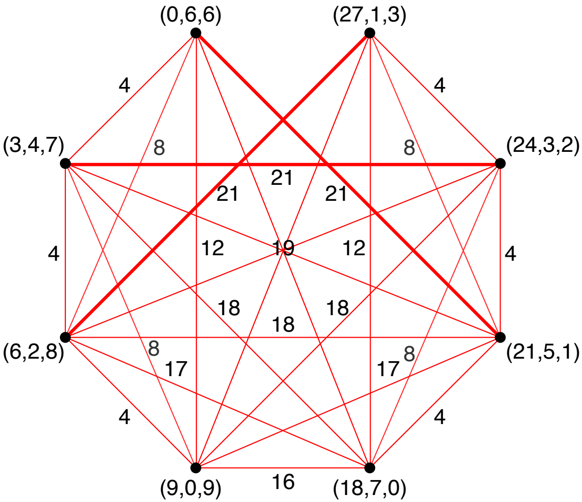

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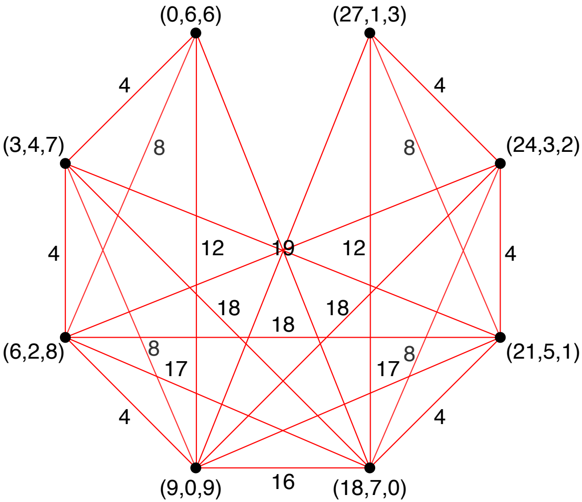

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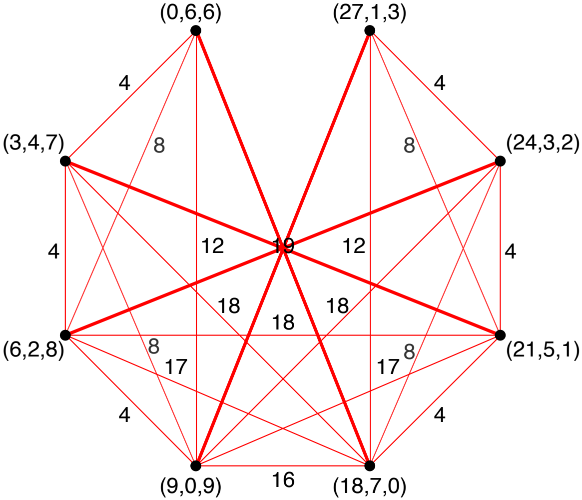

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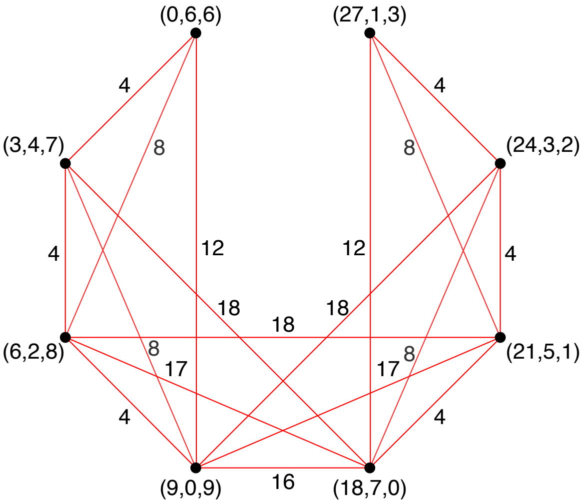

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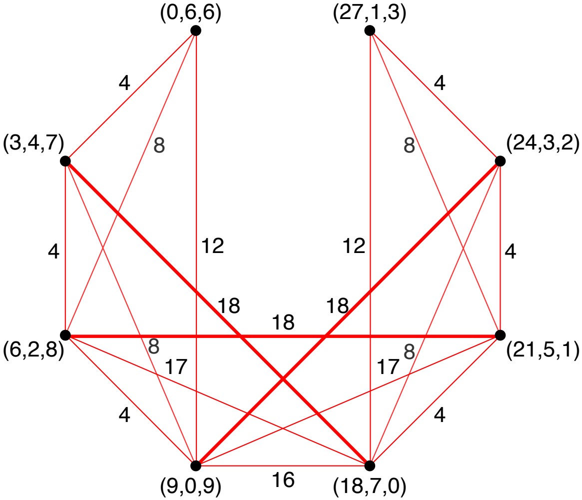

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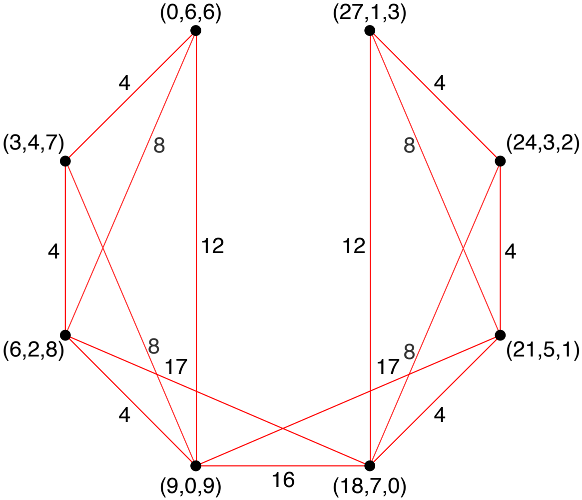

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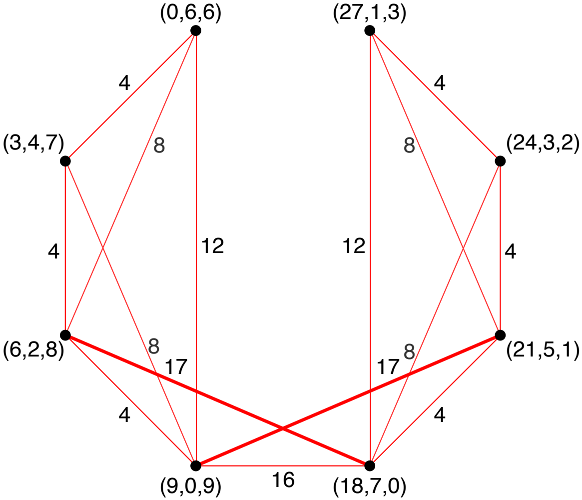

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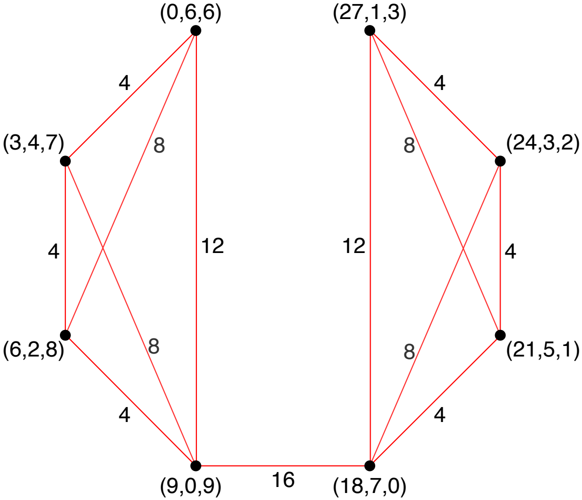

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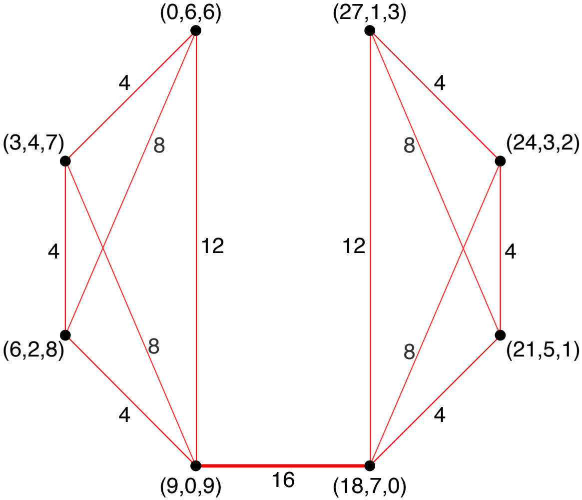

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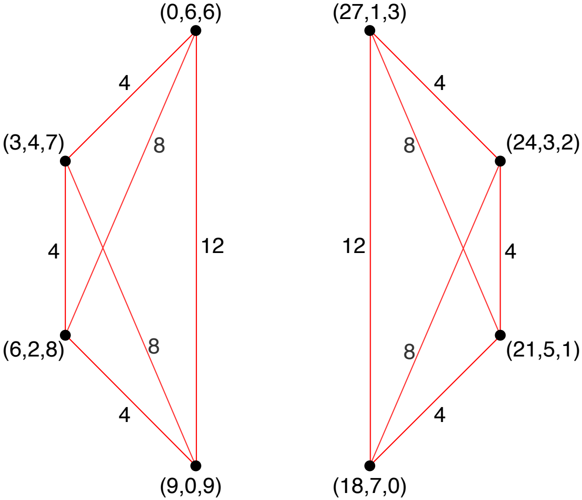

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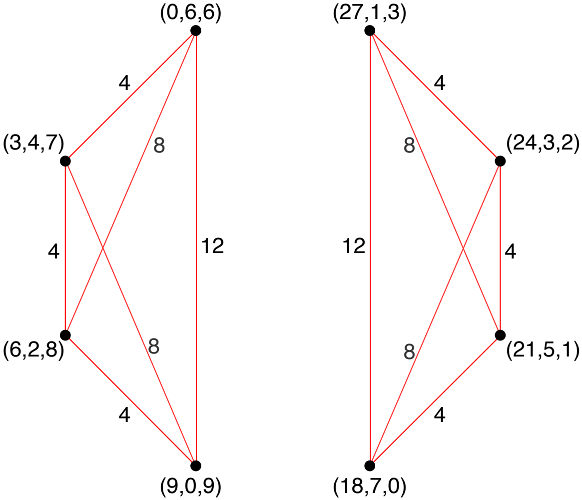
$S = \langle 11, 36, 39 \rangle, n = 450,\ c(n) = 16$



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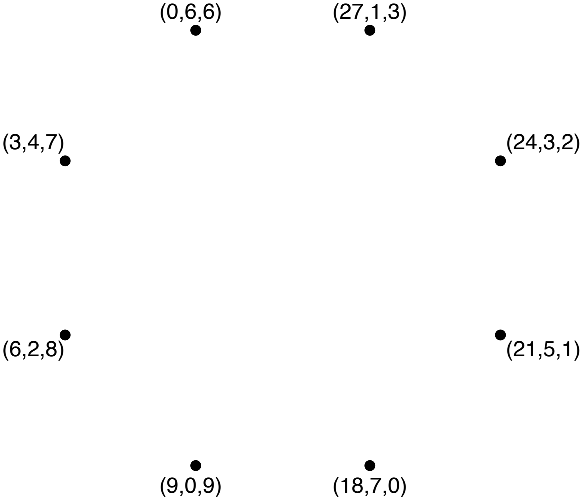

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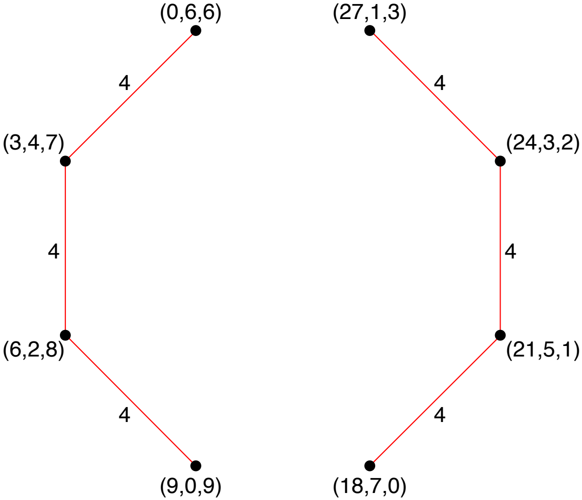

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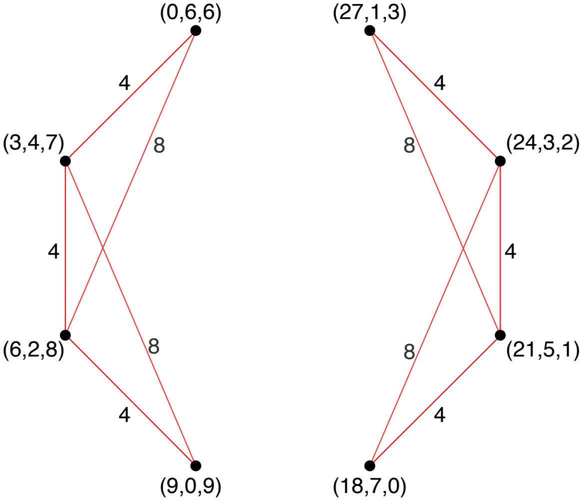

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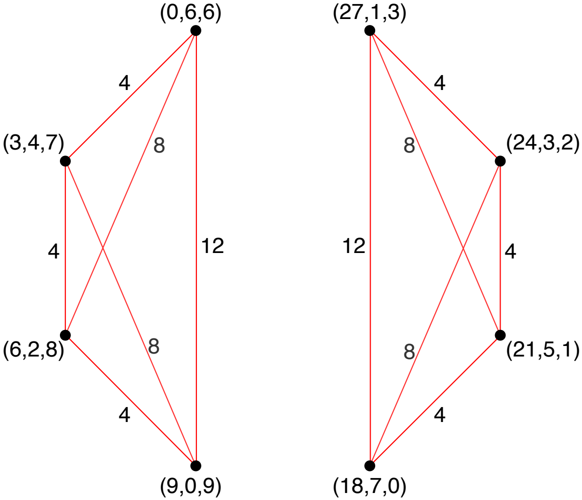

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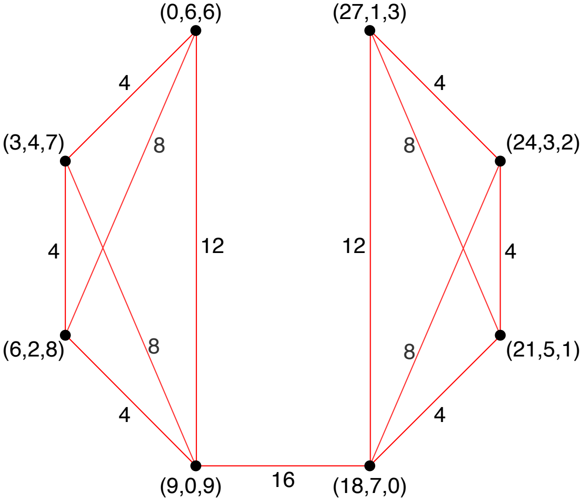

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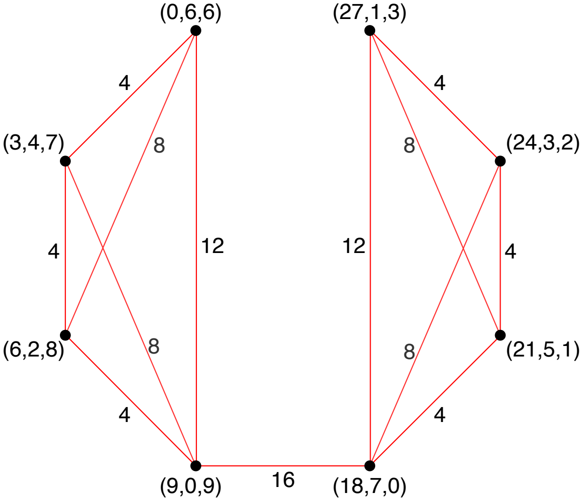

Betti elements

Definition

For an element $n \in S = \langle n_1, \ldots, n_k \rangle$, let ∇_n denote the subgraph of the catenary graph in which only edges (f, f') with $\gcd(f, f') \neq 0$ are drawn.

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$S = \langle 10, 15, 17 \rangle$ has Betti elements 30 and 85.

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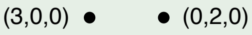
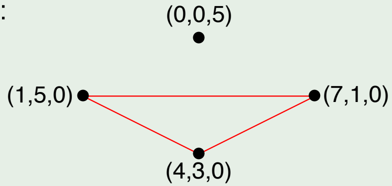
∇_{30} :



∇_{85} :



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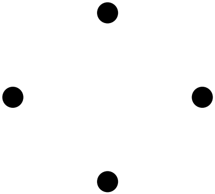
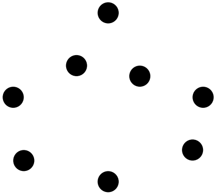


$\mathsf{Z}_S(n)$



$\mathsf{Z}_S(n + n_i)$

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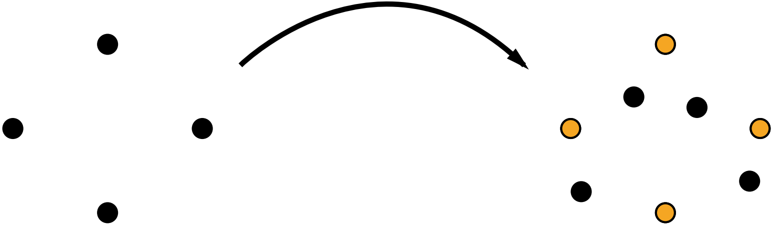

$$\mathsf{Z}_S(n) \hookrightarrow \mathsf{Z}_S(n + n_i)$$

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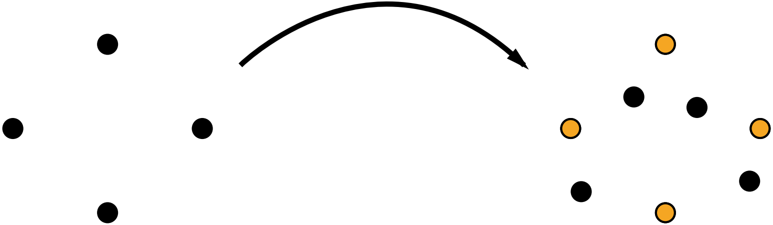

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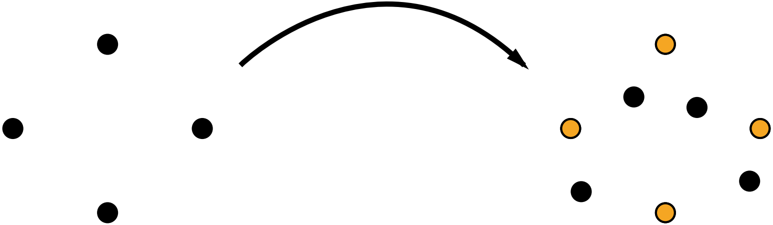

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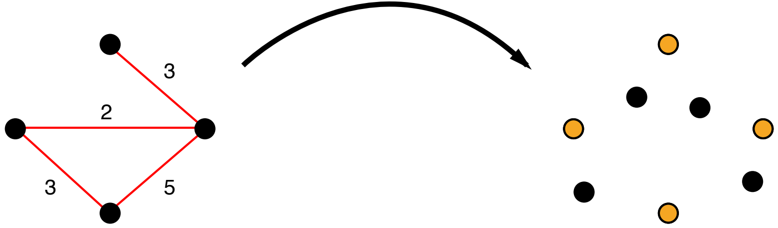

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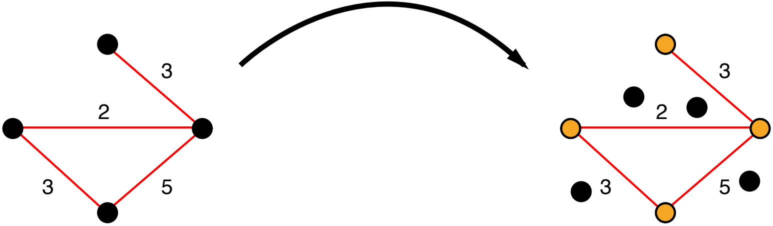

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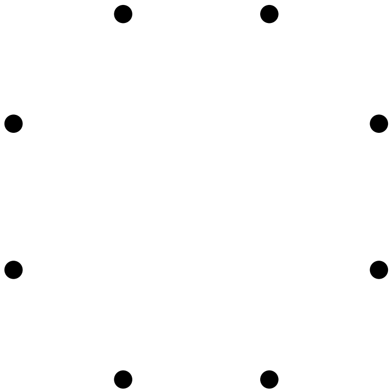

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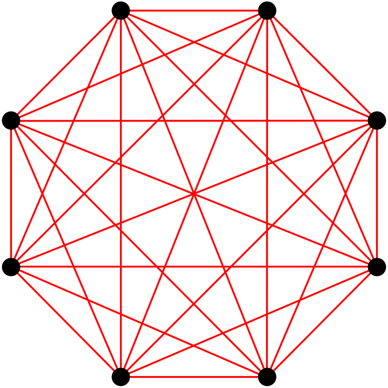

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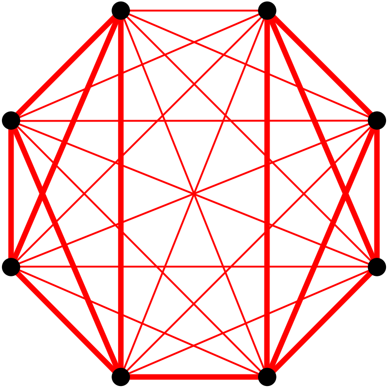
Idea for proof: The catenary graph of each $n \in S$ is "spanned" by certain edges determined by Betti elements.



Minimal (nonzero) catenary degree in S

Conjecture

$\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}$.

Minimal (nonzero) catenary degree in S

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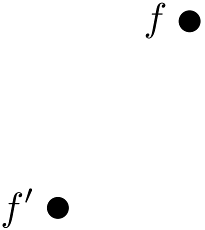

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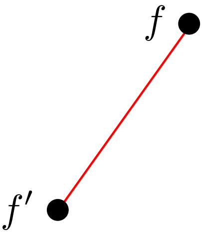

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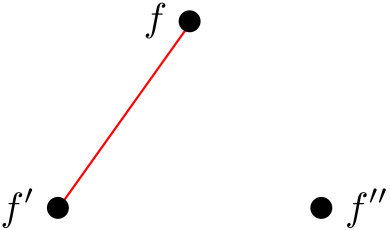

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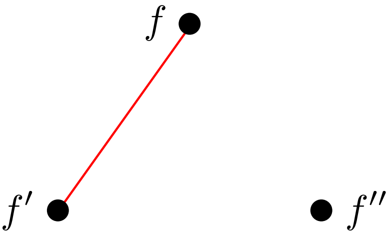

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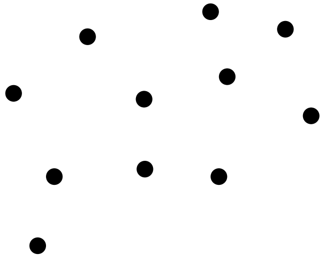

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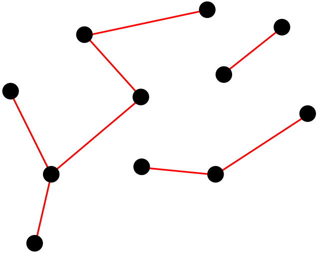

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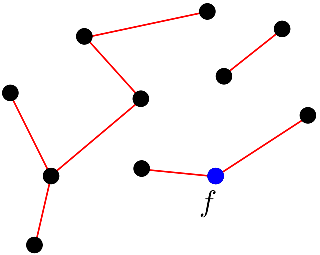

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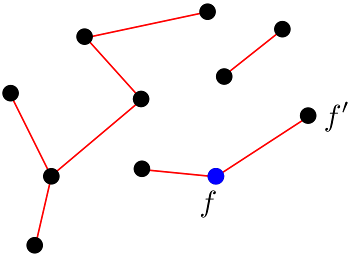

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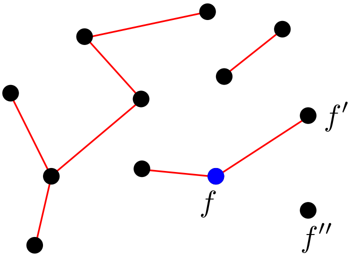

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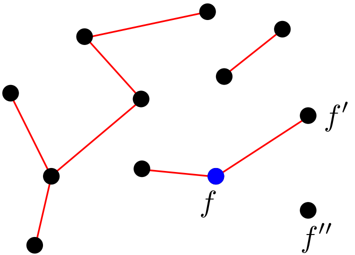
- $f' \in \mathsf{Z}_S(n)$ with $d(f, f') < B$
- Lemma $\Rightarrow |f''| > |f|$
- maximality of $|f| \Rightarrow$
  f'' has no edges!

Catenary graph of n:



Switching gears: ω-primality

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Definition (ω-primality)

Fix a cancellative, commutative, atomic monoid M. For $x \in M$, $\omega(x)$ is the smallest positive integer m such that whenever $x \mid \prod_{i=1}^{r} u_i$ for $r > m$, there exists a subset $T \subset \{1, \ldots, r\}$ with $|T| \le m$ such that $x \mid \prod_{i \in T} u_i$.

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Fact

$\omega(x) = 1$ if and only if x is prime (i.e. $x \mid ab$ implies $x \mid a$ or $x \mid b$).

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M is factorial if and only if every irreducible element $u \in M$ is prime. Moreover, $\omega(p_1 \cdots p_r) = r$ for any primes $p_1, \ldots, p_r \in M$.

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A *bullet* for $x \in M$ is a product $u_1 \cdots u_r$ of irreducible elements such that (i) x divides $u_1 \cdots u_r$, and (ii) x does not divide $u_1 \cdots u_r / u_i$ for each $i \le r$. The set of bullets of x is denoted $\mathrm{bul}(x)$.

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$\omega_M(x) = \max\{r : u_1 \cdots u_r \in \mathrm{bul}(x)\}$.

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Remark

Several improvements on this algorithm exist.

Quasilinearity for numerical monoids

Theorem ((O.–Pelayo, 2013), (García-García et.al., 2013))

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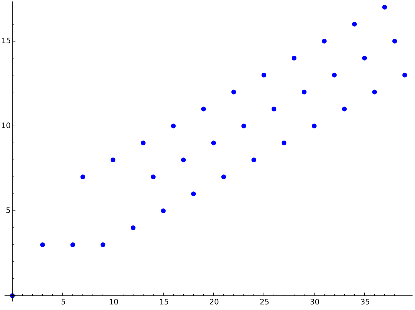
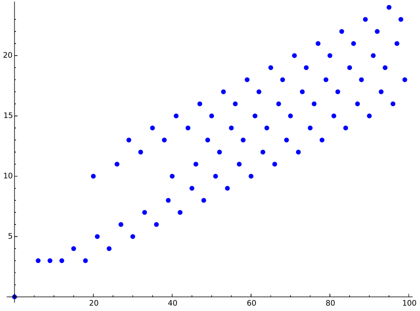


$S = \langle 3, 7 \rangle$



$McN = \langle 6, 9, 20 \rangle$

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Answer (Barron-O.-Pelayo, 2014)

Yes!

Toward a dynamic algorithm. . . the inductive step

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Moreover, $\mathsf{bul}(n) = \bigcup_{i \leq k} \psi_i(\mathsf{bul}(n - n_i))$.**

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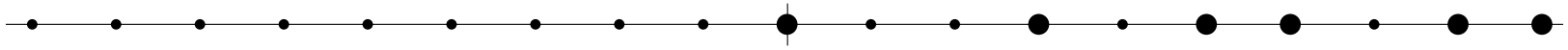

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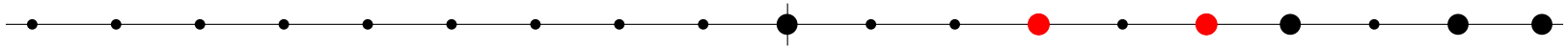

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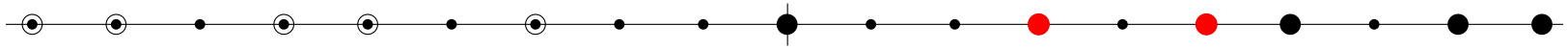

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Remark

All properties of ω extend from S to $\mathbb{Z}$.

Proposition

For $n \in \mathbb{Z}$, the following are equivalent:
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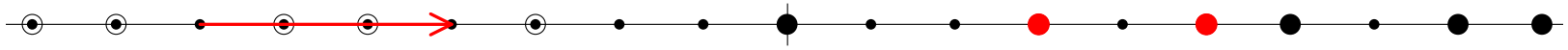

Toward a dynamic algorithm. . . the base case

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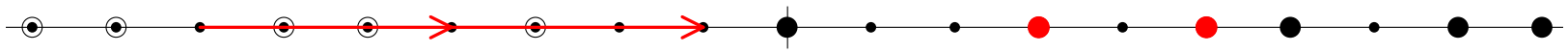

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6	3	$\{3\vec{e}_3, 2\vec{e}_2, \vec{e}_1\}$
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$\langle 11, 13, 15 \rangle$	10000	915	———	42ms
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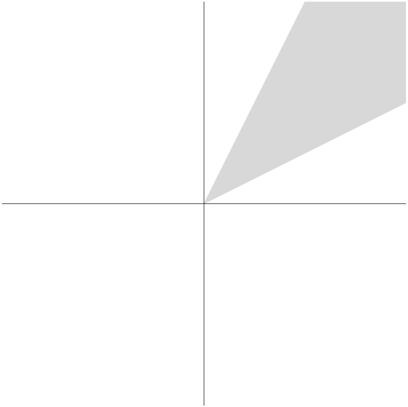

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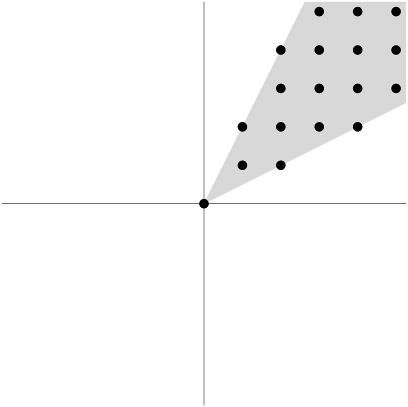

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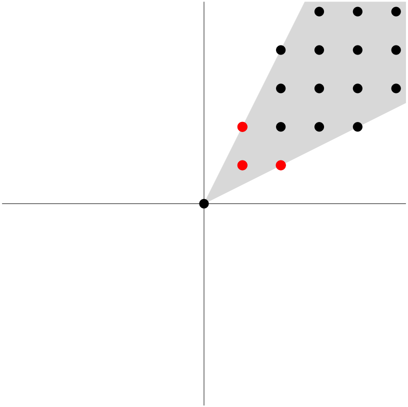

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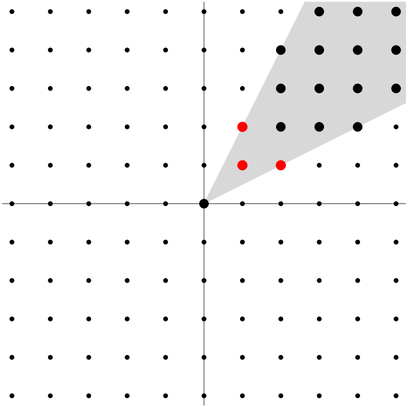

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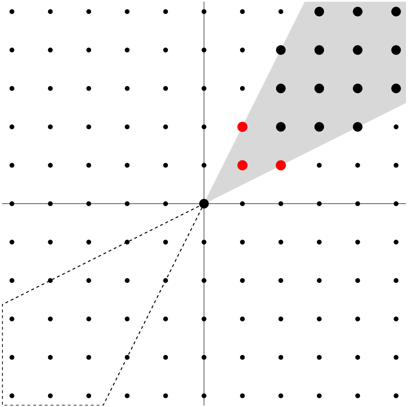

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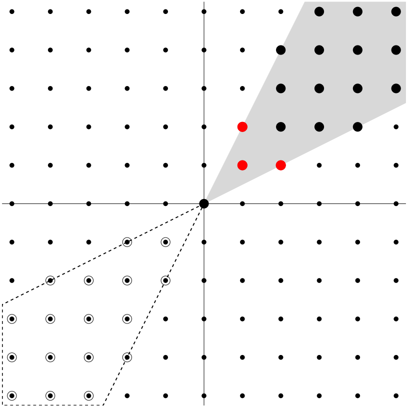

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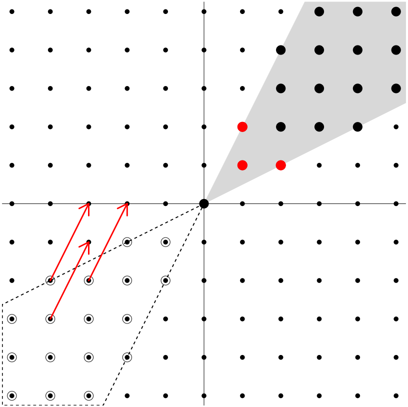

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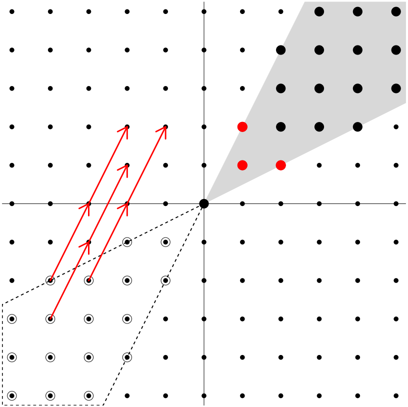

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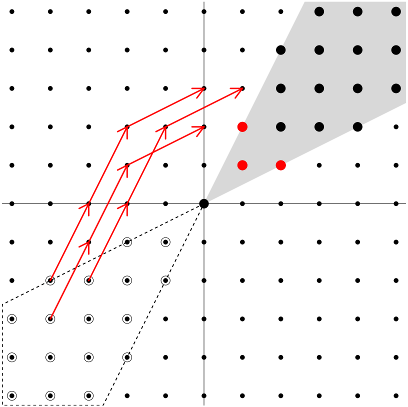

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Problem

Find a dynamic algorithm to compute catenary degrees.

References

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