Invariants of non-unique factorization

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November 21, 2014

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First half: Catenary degree (combinatorial)
Joint with Vadim Ponomarenko, Reuben Tate*, and Gautam Webb*

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Joint with Vadim Ponomarenko, Reuben Tate*, and Gautam Webb*

Second half: ω-primality (algebraic)
Joint with Thomas Barron* and Roberto Pelayo

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Factorial domains

Definition

An integral domain R is *factorial* if for each non-unit $r \in R$,

1. there is a *factorization* $r = u_1 \cdots u_k$ as a product of irreducibles, and
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The point: it's complicated.

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$$\begin{array}{rclcc} 60 & = & 7(6) + 2(9) & \rightsquigarrow & (7, 2, 0) \\ & = & 3(20) & \rightsquigarrow & (0, 0, 3) \end{array}$$

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$$\mathsf{Z}_S(n) = \left\{(a_1, \ldots, a_k) \in \mathbb{N}^k : n = a_1 n_1 + \cdots + a_k n_k\right\}$$

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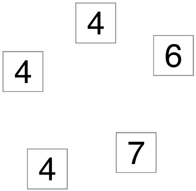

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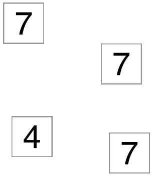

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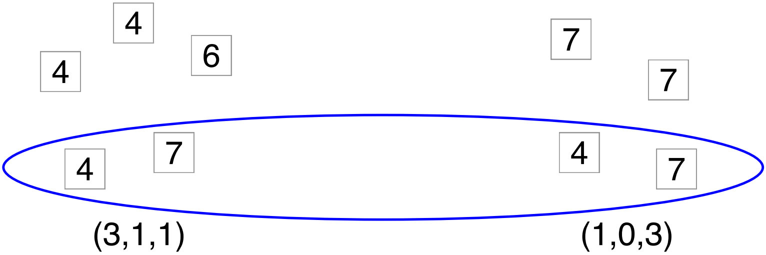

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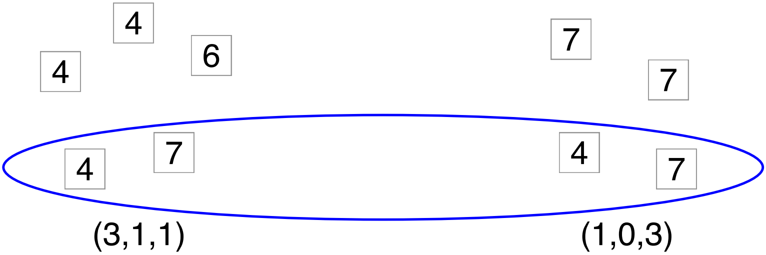

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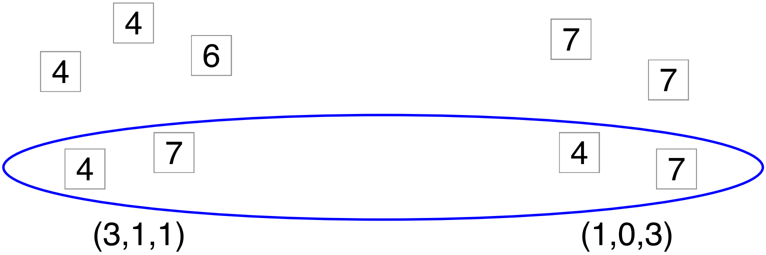

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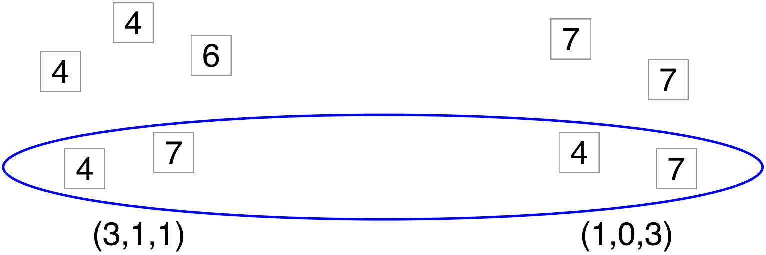

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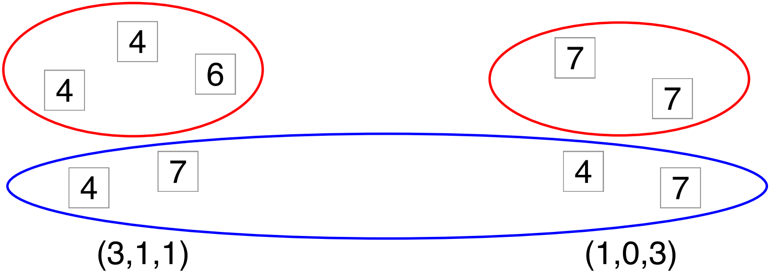

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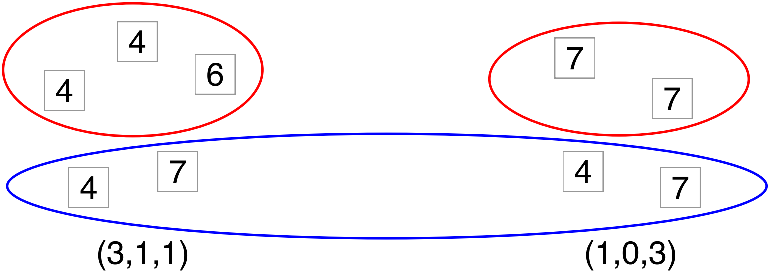

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If $|\mathsf{Z}_S(n)| = 1$, define $c(n) = 0$.

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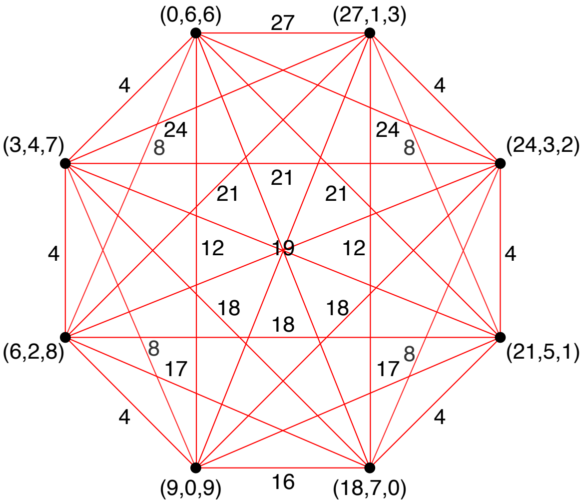

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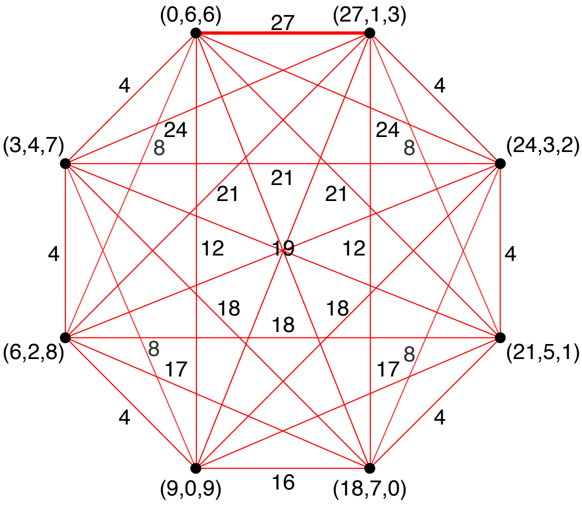

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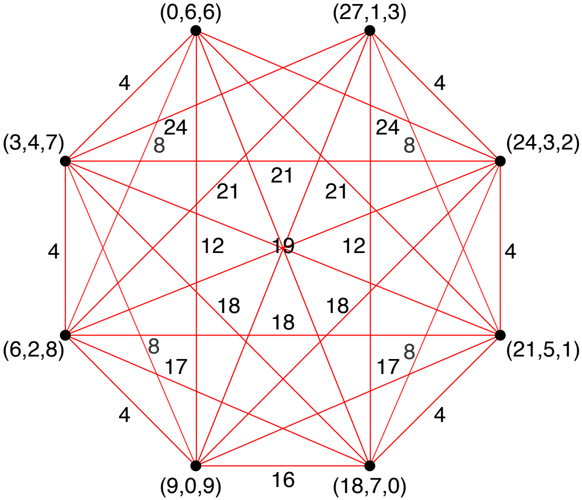

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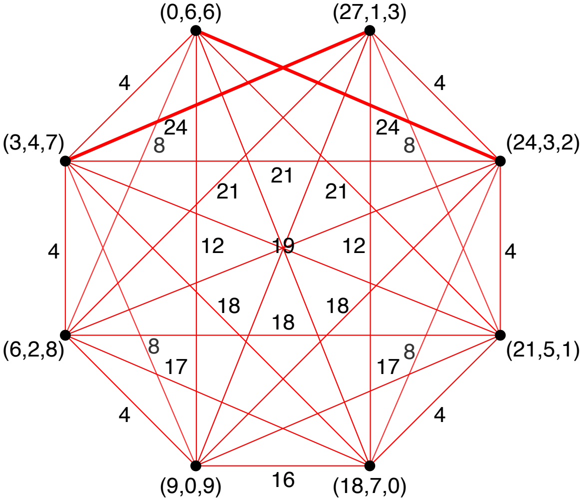

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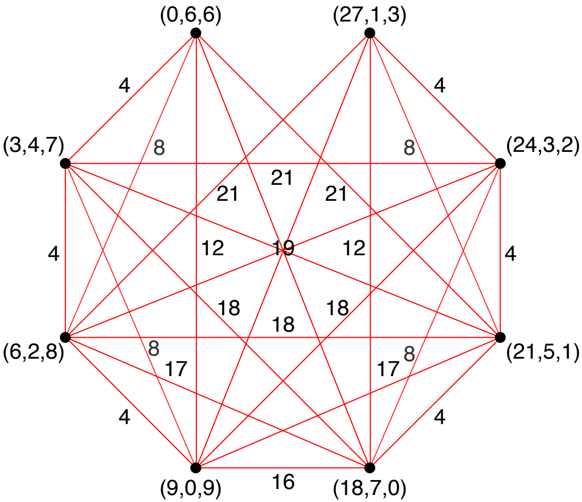

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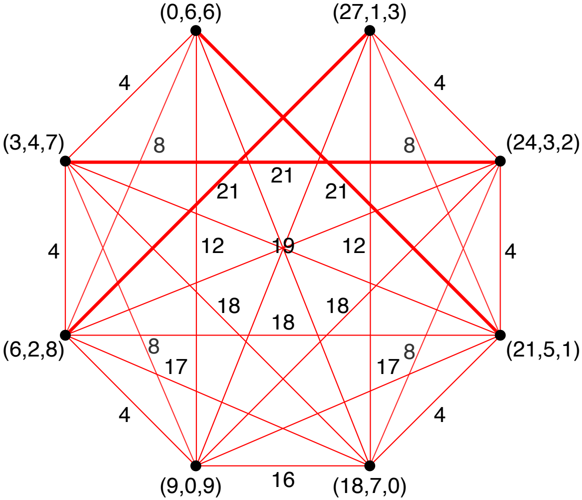

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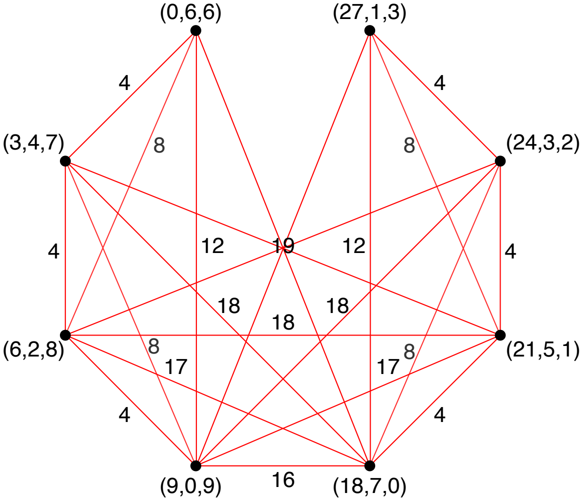

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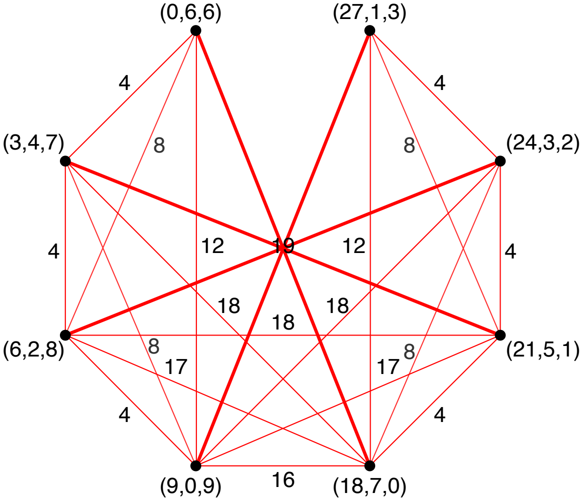

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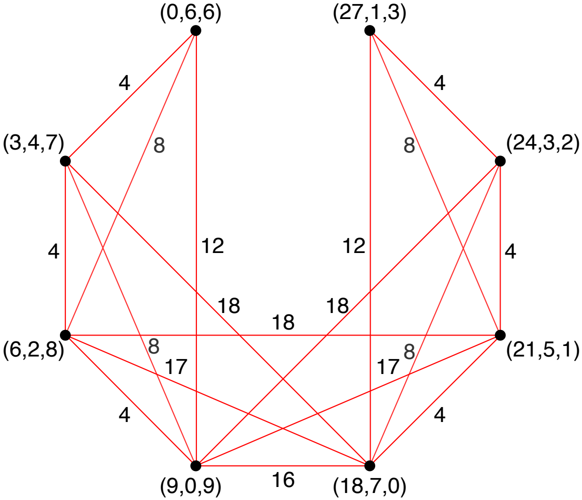

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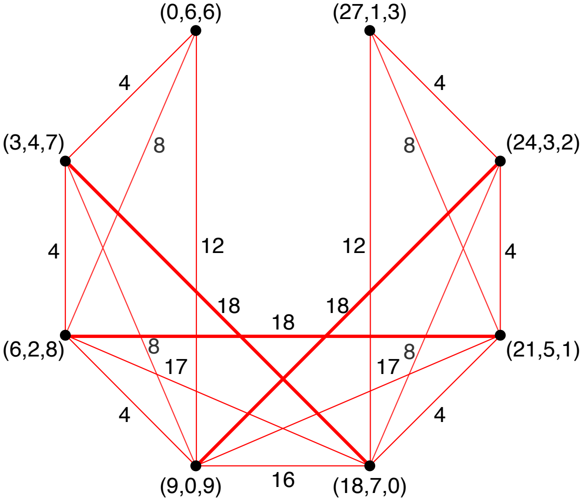

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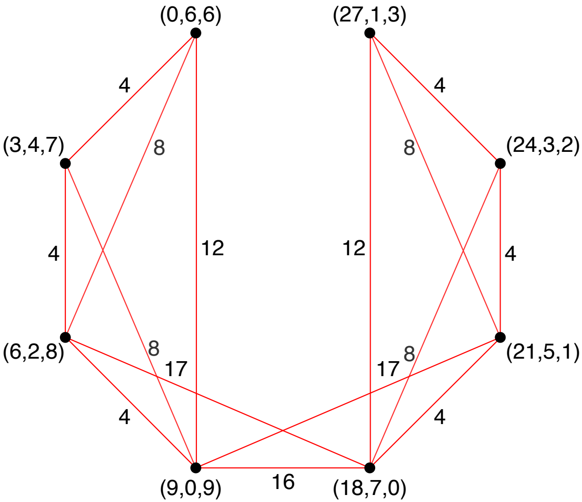

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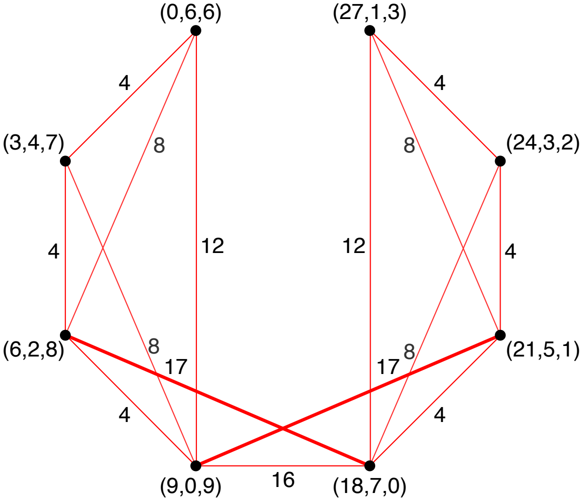

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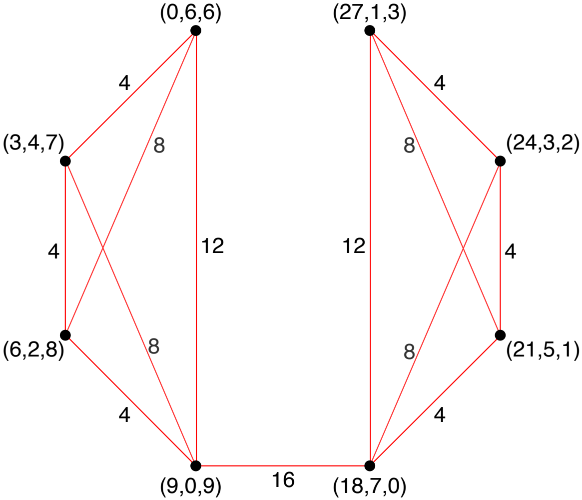

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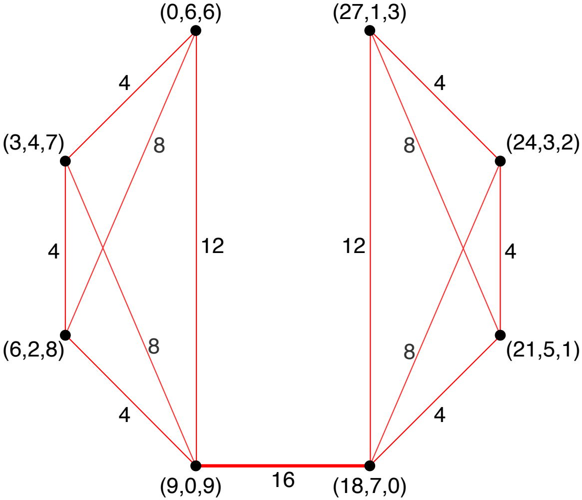

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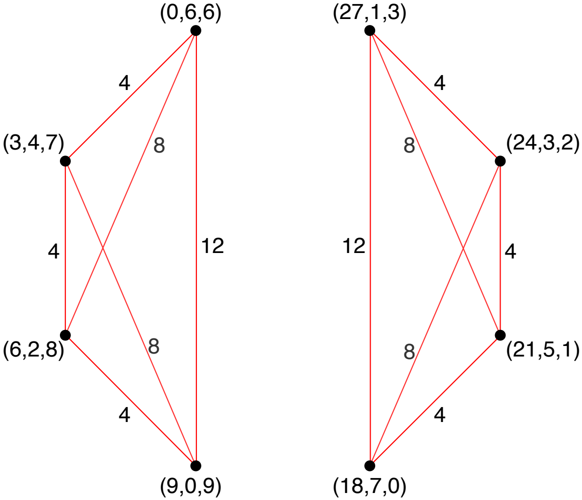

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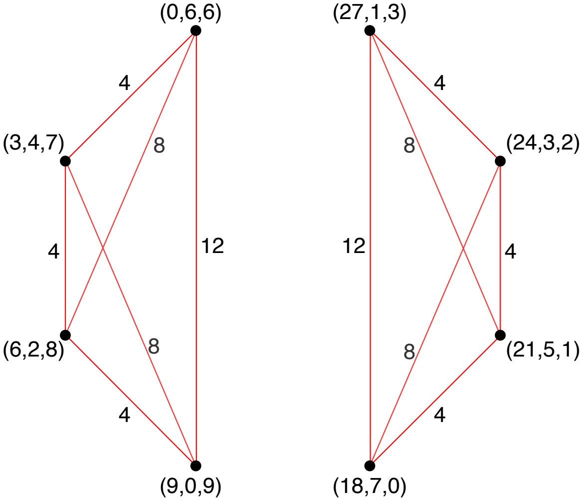
$S = \langle 11, 36, 39 \rangle, n = 450, \ c(n) = 16$



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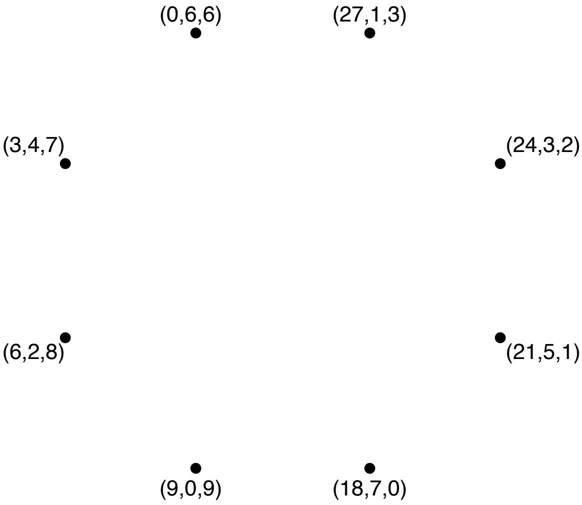

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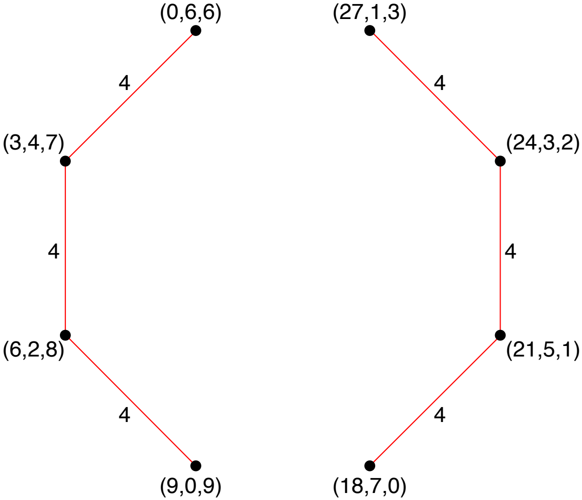

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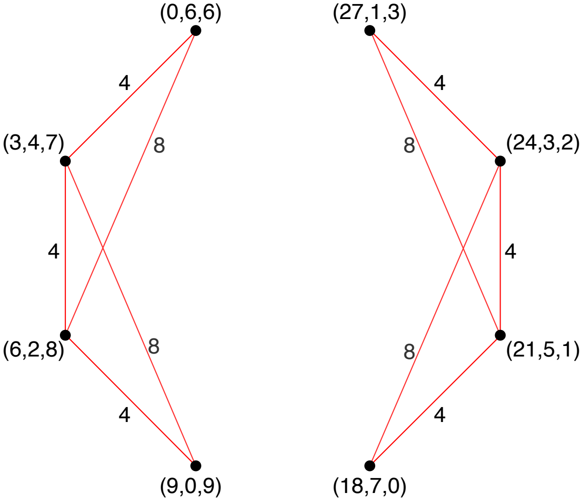

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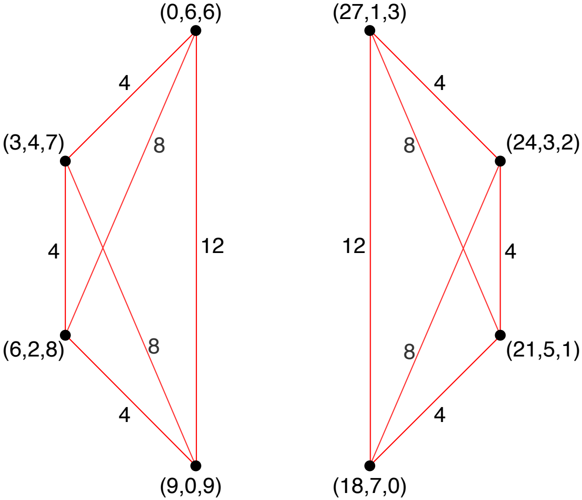

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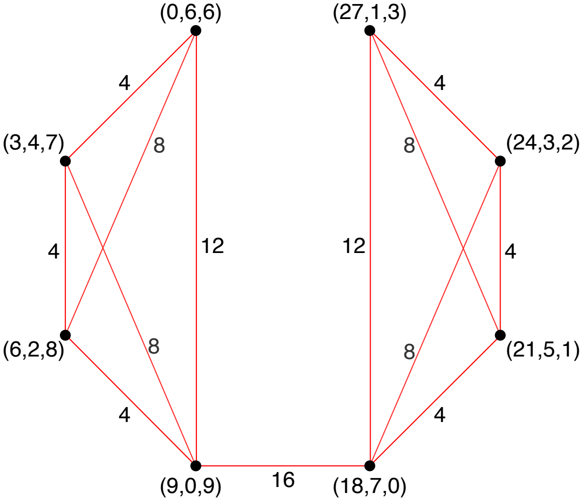

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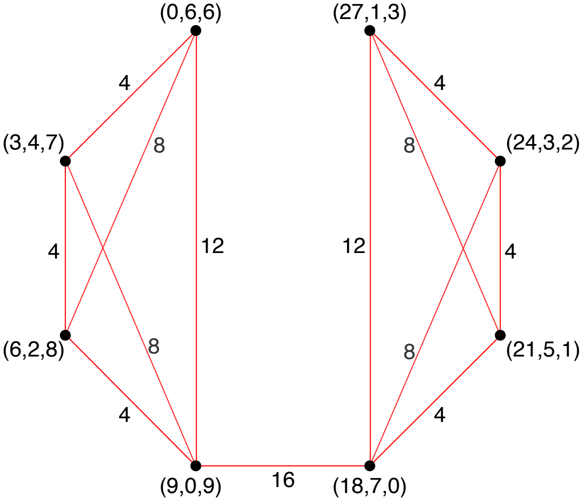

Betti elements

Definition

For an element $n \in S = \langle n_1, \ldots, n_k \rangle$, let ∇_n denote the subgraph of the catenary graph in which only edges (f, f') with $\gcd(f, f') \neq 0$ are drawn.

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$S = \langle 10, 15, 17 \rangle$ has Betti elements 30 and 85.

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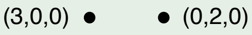

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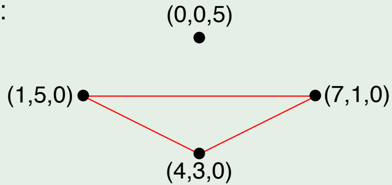

Maximal catenary degree in S

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$\max\{c(n) : n \in S\} = \max\{c(b) : b$ Betti element of $S\}$.

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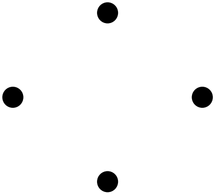
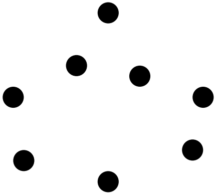


$Z_S(n)$



$Z_S(n + n_i)$

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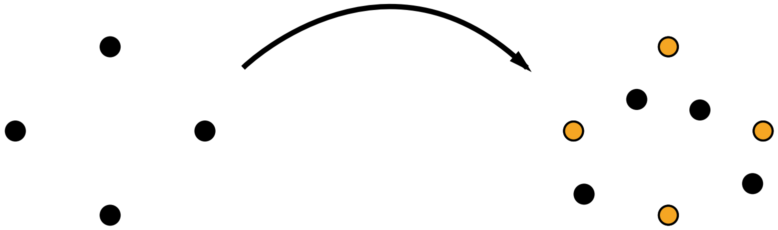

$$\mathsf{Z}_S(n) \hookrightarrow \mathsf{Z}_S(n + n_i)$$

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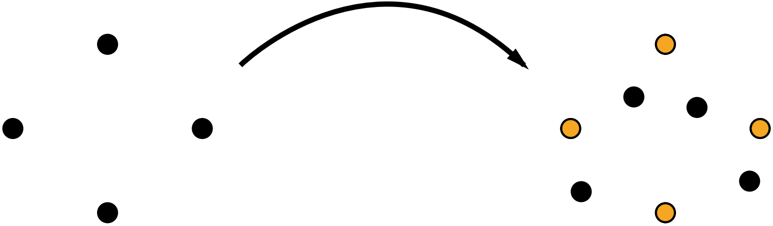

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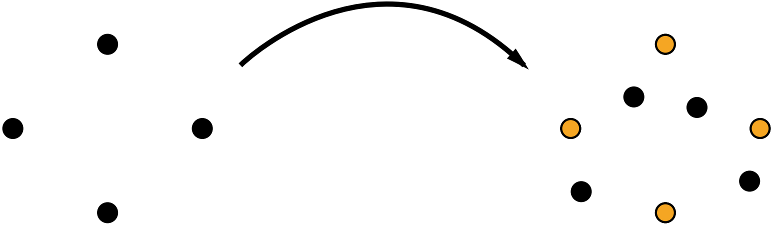

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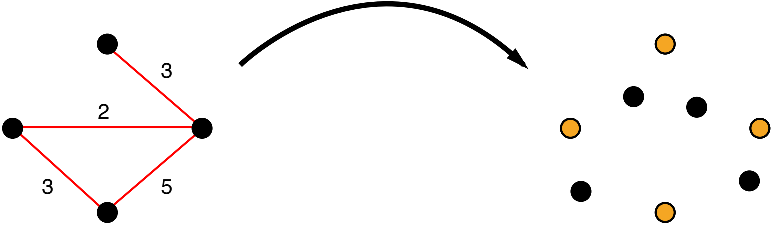

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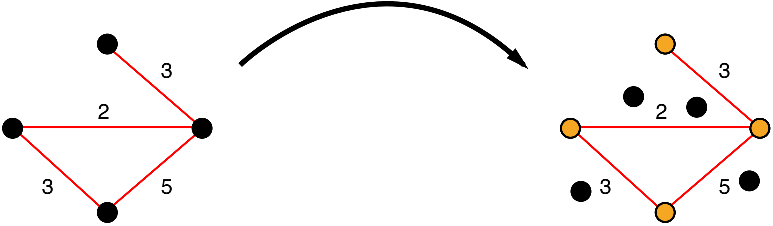

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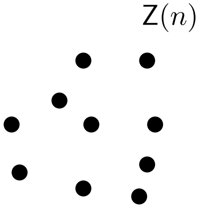

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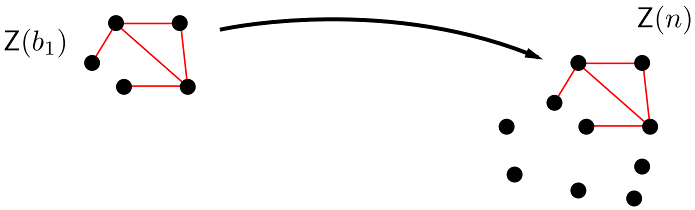

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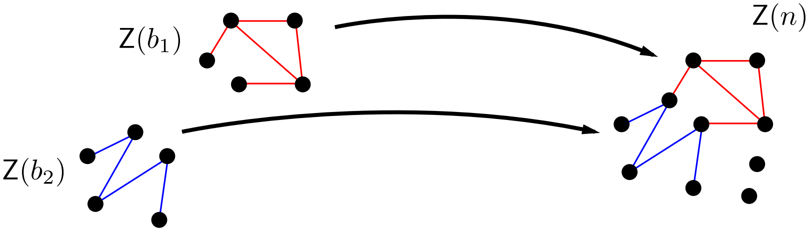

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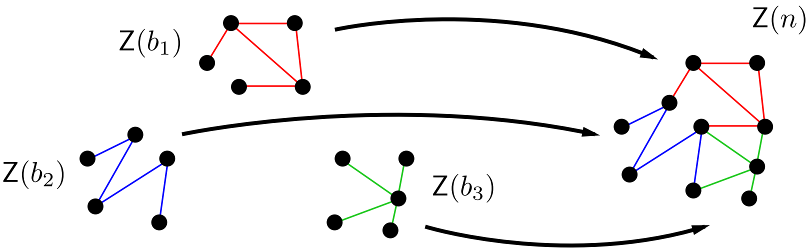

Minimal (nonzero) catenary degree in S

Conjecture

$\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}$.

Minimal (nonzero) catenary degree in S

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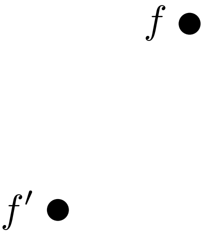

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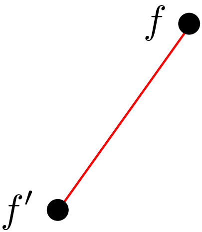

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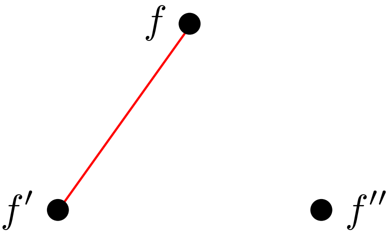

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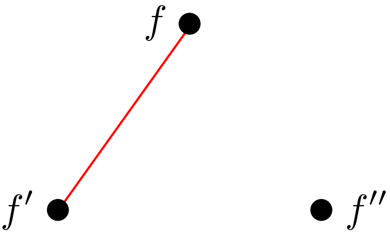

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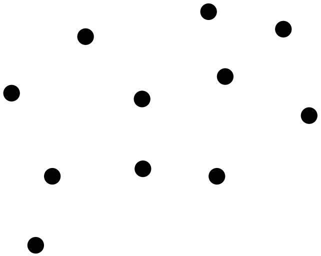

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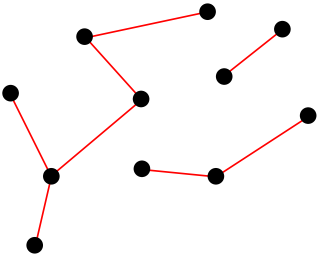

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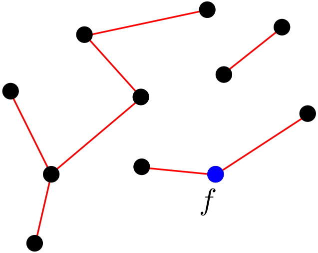

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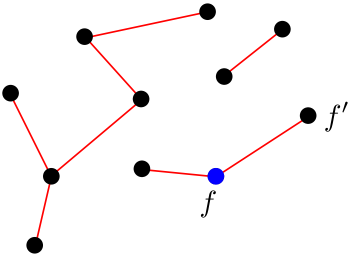

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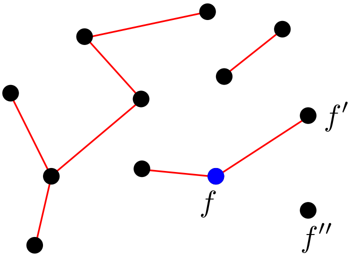

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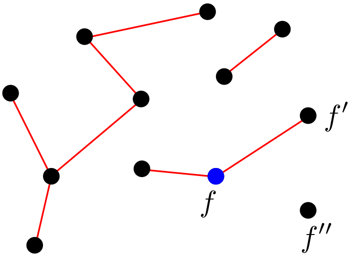
- $f' \in \mathsf{Z}_S(n)$ with $d(f, f') < B$
- Lemma $\Rightarrow |f''| > |f|$
- maximality of $|f| \Rightarrow$
  f'' has no edges!

Catenary graph of n:



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Fix a cancellative, commutative, atomic monoid M. For $x \in M$, $\omega(x)$ is the smallest positive integer m such that whenever $x \mid \prod_{i=1}^{r} u_i$ for $r > m$, there exists a subset $T \subset \{1, \ldots, r\}$ with $|T| \le m$ such that $x \mid \prod_{i \in T} u_i$.

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$\omega(x) = 1$ if and only if x is prime (i.e. $x \mid ab$ implies $x \mid a$ or $x \mid b$).

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M is factorial if and only if every irreducible element $u \in M$ is prime. Moreover, $\omega(p_1 \cdots p_r) = r$ for any primes $p_1, \ldots, p_r \in M$.

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A *bullet* for $x \in M$ is a product $u_1 \cdots u_r$ of irreducible elements such that (i) x divides $u_1 \cdots u_r$, and (ii) x does not divide $u_1 \cdots u_r / u_i$ for each $i \le r$. The set of bullets of x is denoted $\mathrm{bul}(x)$.

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Proposition

$\omega_M(x) = \max\{r : u_1 \cdots u_r \in \mathrm{bul}(x)\}$.

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Remark

Several improvements on this algorithm exist.

Quasilinearity for numerical monoids

Theorem ((O.–Pelayo, 2013), (García-García et.al., 2013))

$\omega_S(n) = \frac{1}{n_1}n + a_0(n)$ *for* $n \gg 0$, *where* $a_0(n)$ *periodic with period* n_1.

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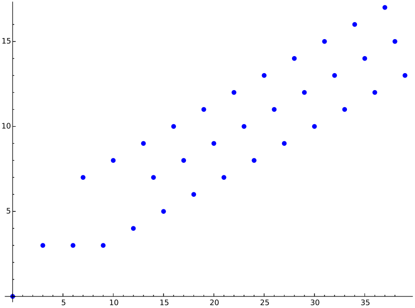
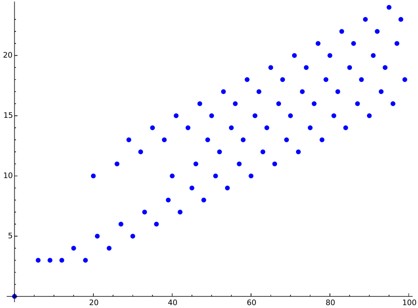


$S = \langle 3, 7 \rangle$



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Answer (Barron-O.-Pelayo, 2014)

Yes!

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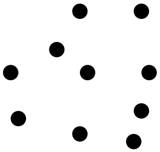

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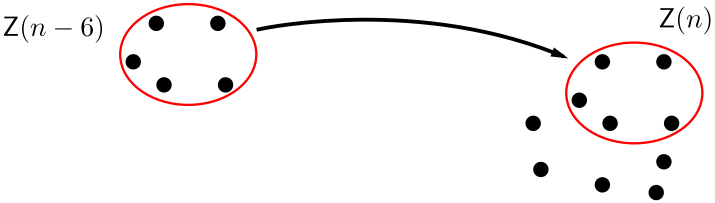

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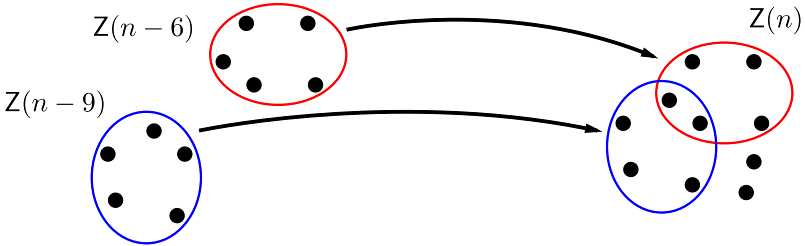

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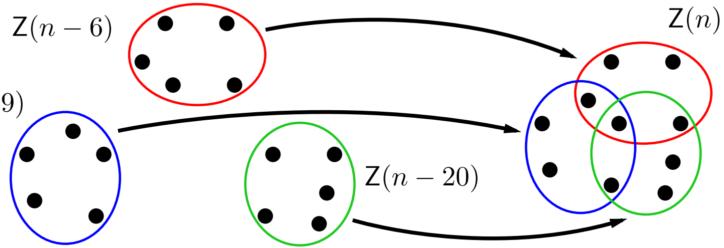

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Definition/Proposition (Cover morphisms)

Fix $n \in S$ and $i \leq k$. The *i-th cover morphism for n* is the map

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$$\begin{array}{rcl} \phi_i : \mathsf{Z}(n - n_i) & \longrightarrow & \mathsf{Z}(n) \\ \vec{a} & \longmapsto & \vec{a} + \vec{e}_i. \end{array}$$

In particular,

$$\mathsf{Z}(n) = \bigcup_{i \leq k} \phi_i(\mathsf{Z}(n - n_i))$$

Definition/Proposition (Cover morphisms)

Fix $n \in S$ and $i \leq k$. The *i-th cover morphism for n* is the map

$$\psi_i : \mathrm{bul}(n - n_i) \longrightarrow \mathrm{bul}(n)$$

given by

$$\vec{b} \longmapsto \begin{cases} \vec{b} + e_i & \sum_{j=1}^k b_j n_j - n - n_i \notin S \\ \vec{b} & \sum_{j=1}^k b_j n_j - n - n_i \in S \end{cases}$$

Toward a dynamic algorithm... the inductive step

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Moreover, $\mathsf{bul}(n) = \bigcup_{i \le k} \psi_i(\mathsf{bul}(n - n_i))$.**

Toward a dynamic algorithm…the base case

Definition (ω-primality in numerical monoids)

Fix a numerical monoid S and $n \in S$.

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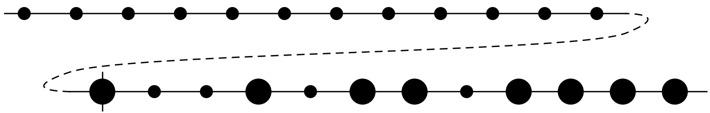

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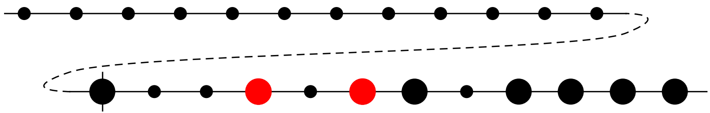

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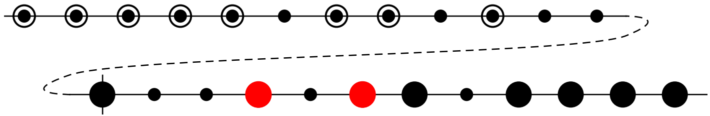

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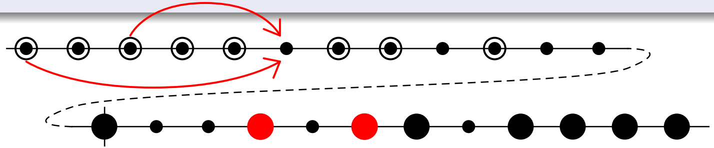

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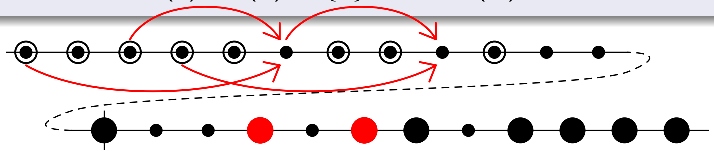

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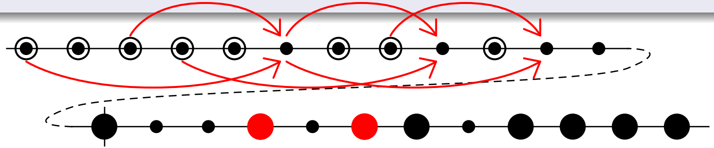

A dynamic algorithm!

Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$.

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-34	2	$\{\vec{e}_1, 2\vec{e}_2, \vec{e}_3\}$
-33	0	$\{\vec{0}\}$
-32	0	$\{\vec{0}\}$
-31	3	$\{3\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

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$\vdots$	$\vdots$	$\vdots$

$n \in \mathbb{Z}$	$\omega(n)$	$\mathrm{bul}(n)$
6	3	$\{3\vec{e}_3, 2\vec{e}_2, \vec{e}_1\}$
7	6	$\{6\vec{e}_1, 3\vec{e}_2, 2\vec{e}_3, (3, 1, 0)\}$
8	8	$\{8\vec{e}_1, 6\vec{e}_2, (5, 2, 0), \ldots\}$
9	3	$\{3\vec{e}_1, 3\vec{e}_3, \vec{e}_2\}$
10	5	$\{5\vec{e}_1, 4\vec{e}_2, 2\vec{e}_3, (2, 2, 0)\}$
11	10	$\{10\vec{e}_1, 7\vec{e}_2, (7, 1, 0), \ldots\}$
12	3	$\{3\vec{e}_3, 2\vec{e}_1, 2\vec{e}_2\}$
13	7	$\{7\vec{e}_1, 5\vec{e}_2, (4, 1, 0), \ldots\}$
14	9	$\{9\vec{e}_1, 6\vec{e}_2, (6, 2, 0), \ldots\}$
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149	33	$\{33\vec{e}_1, \ldots\}$
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S	$n \in S$	$\omega_S(n)$	Existing	Dynamic
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$\langle 11, 13, 15 \rangle$	1000	97	0m 10.7s	5ms
$\langle 11, 13, 15 \rangle$	3000	279	14m 34.7s	15ms
$\langle 11, 13, 15 \rangle$	10000	915	———	42ms
$\langle 15, 27, 32, 35 \rangle$	1000	69	3m 54.7s	9ms
$\langle 100, 121, 142, 163, 284 \rangle$	25715	308	———	0m 27s
$\langle 1001, 1211, 1421, 1631, 2841 \rangle$	357362	405	———	57m 27s

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GAP Numerical Semigroups Package, available at

`http://www.gap-system.org/Packages/numericalsgps.html`.

Future directions: ω-primality

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Issue: the base case!

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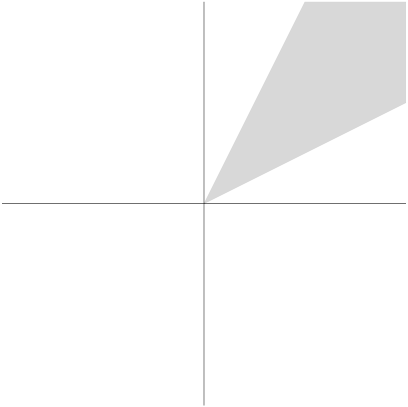

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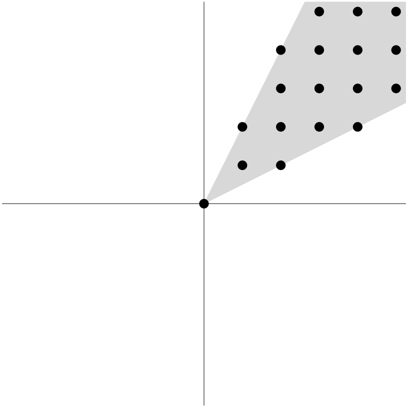

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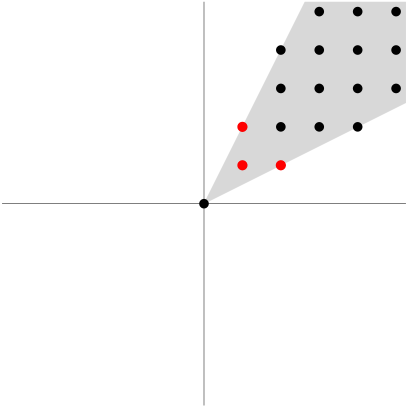

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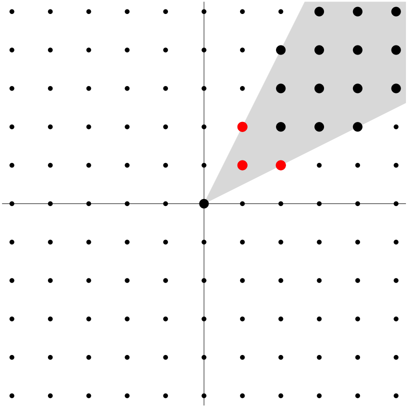

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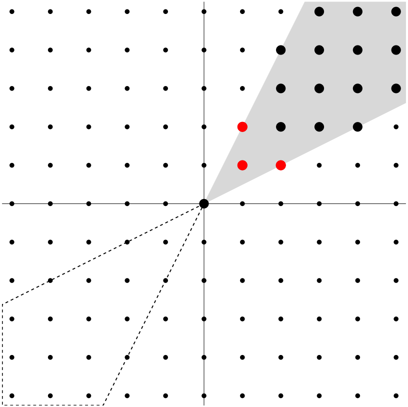

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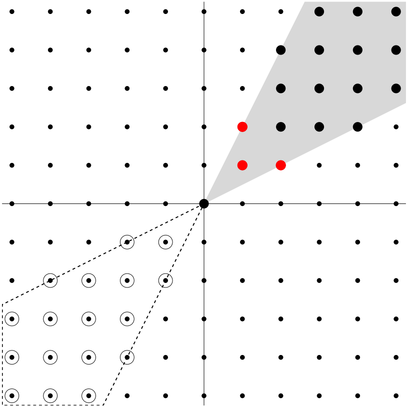

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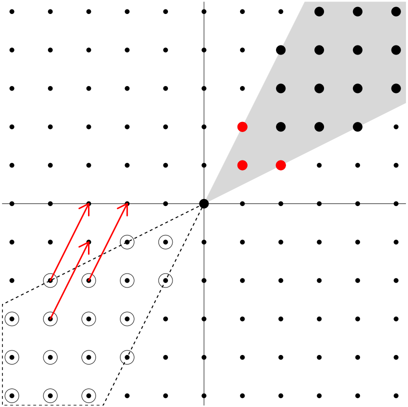

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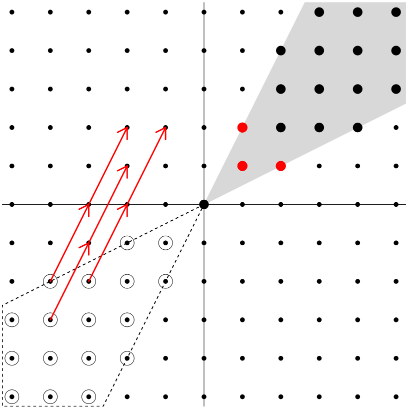

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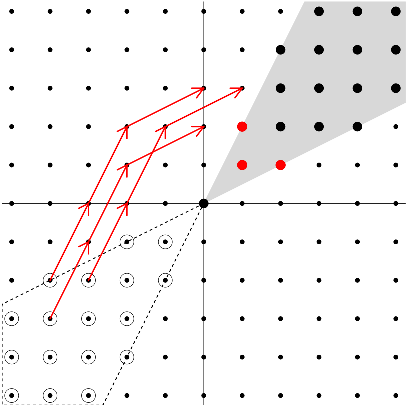

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Find a dynamic algorithm to compute ω-primality in M.

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Future directions: catenary degree

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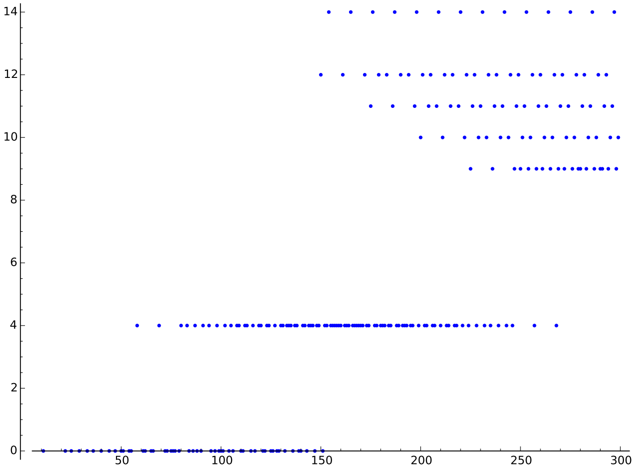

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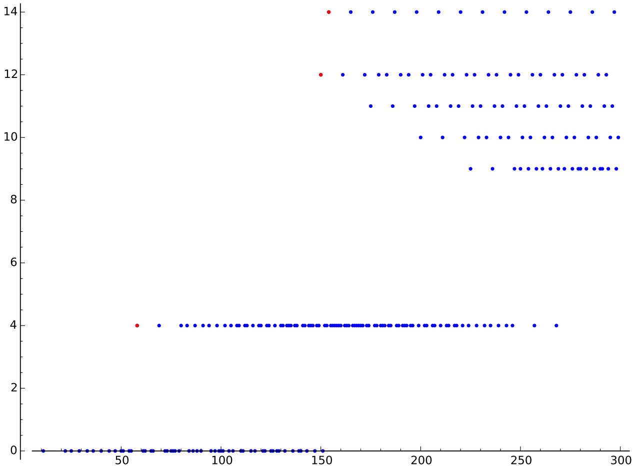

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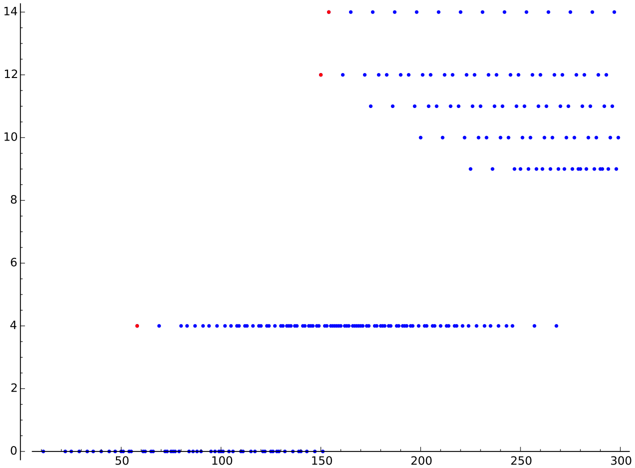

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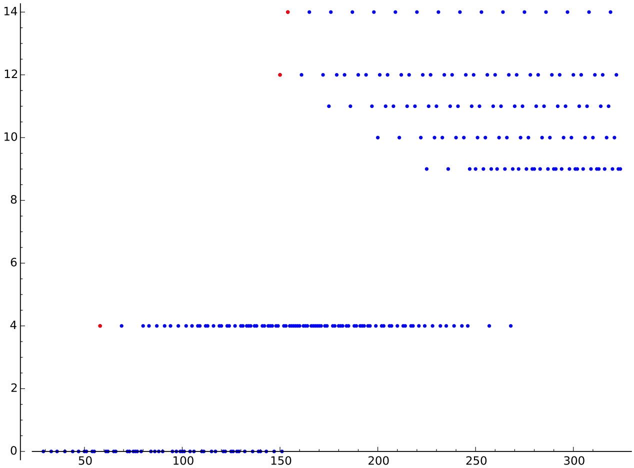

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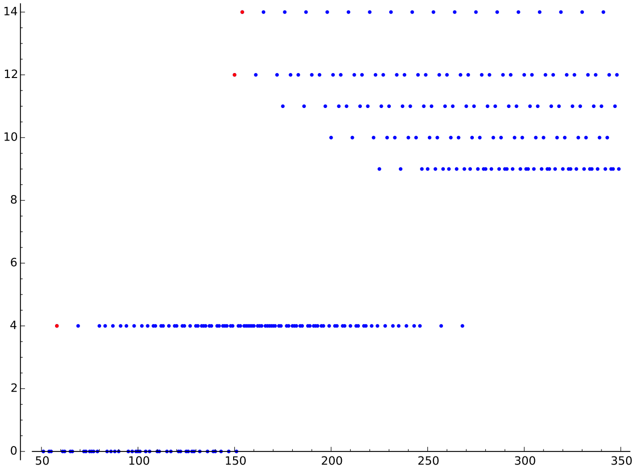

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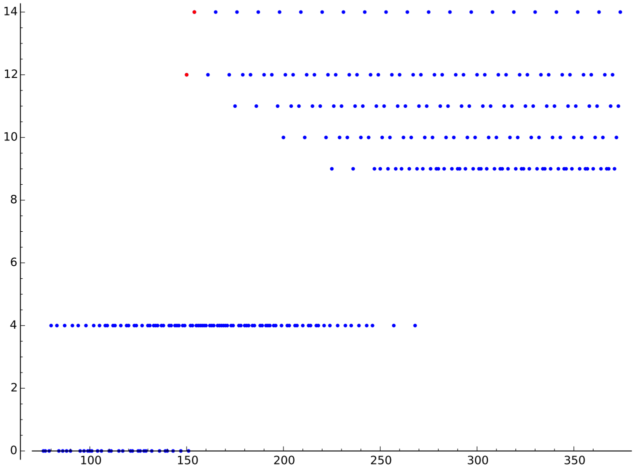

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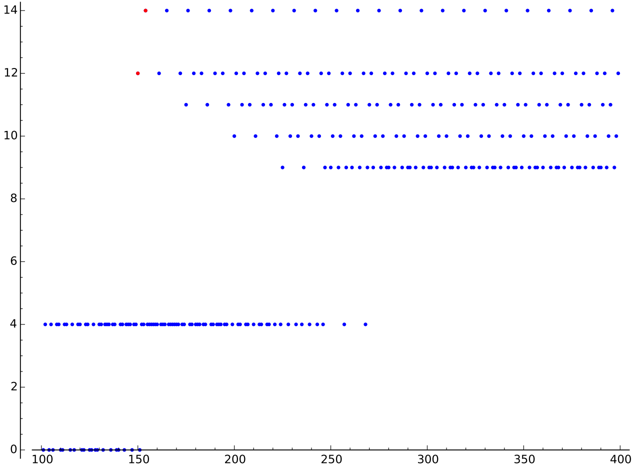

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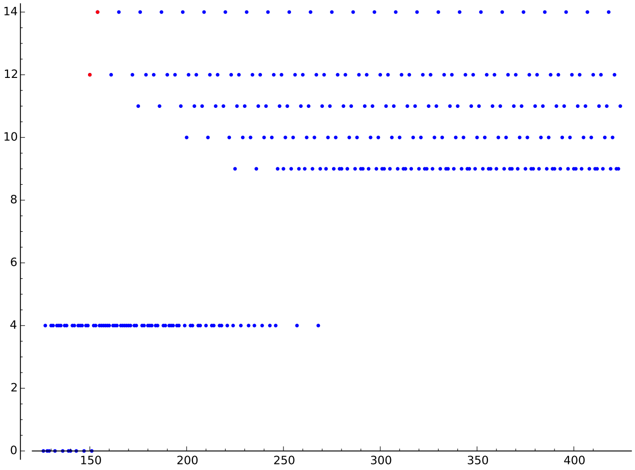

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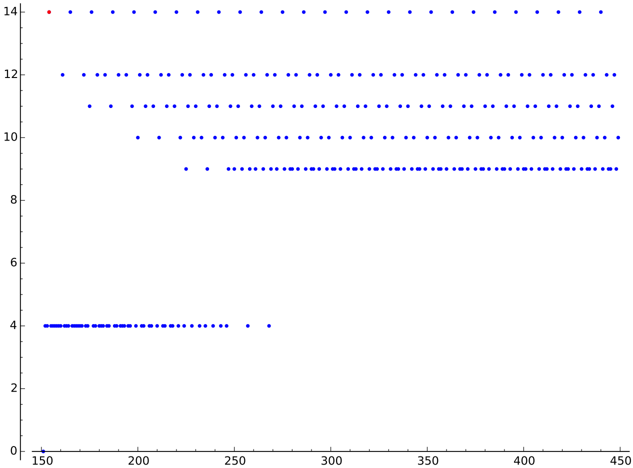

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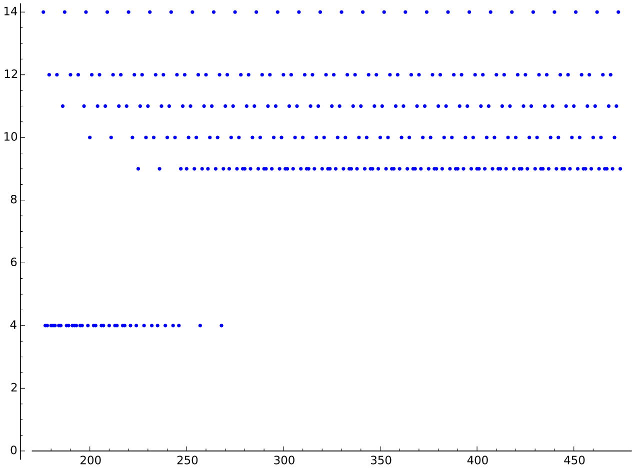

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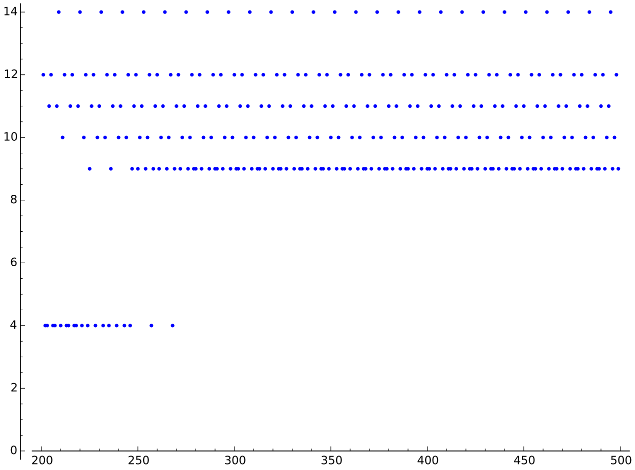

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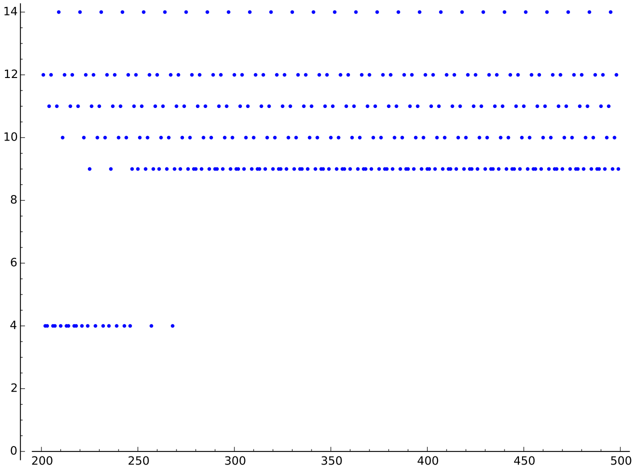

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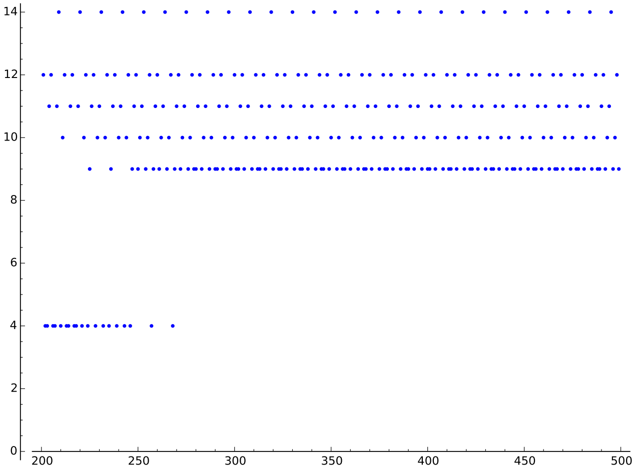

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References

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Chapman & Hall/CRC, Boca Raton, FL, 2006.

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