The set of elasticities in numerical monoids

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Joint with Thomas Barron* and Roberto Pelayo

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A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

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$$\begin{array}{rcllcl} 60 & = & 7(6) + 2(9) & & \rightsquigarrow & (7, 2, 0) \\ & = & & 3(20) & \rightsquigarrow & (0, 0, 3) \end{array}$$

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$$\begin{aligned} \mathsf{Z}_S(n) &= \left\{(a_1, \ldots, a_k) \in \mathbb{N}^k : n = a_1 n_1 + \cdots + a_k n_k\right\}, \\ \mathsf{L}_S(n) &= \{a_1 + \cdots + a_k : (a_1, \ldots, a_k) \in \mathsf{Z}_S(n)\} \end{aligned}$$

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Theorem (J. Amos, S. Chapman, N. Hine, J. Paixão)

Two distinct arithmetical numerical monoids

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satisfy $\mathcal{L}(S) = \mathcal{L}(S')$ if and only if the all of the following hold:

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Corollary

$\mathcal{L}(S)$ does not uniquely determine S as a numerical monoid.

Set of elasticities

Definition

Fix a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$,

$$\rho_S(n) \quad = \quad \max \mathsf{L}_S(n) / \min \mathsf{L}_S(n)$$

denotes the *elasticity* of n, and $R(S) = \{\rho_S(n) : n \in S\}$.

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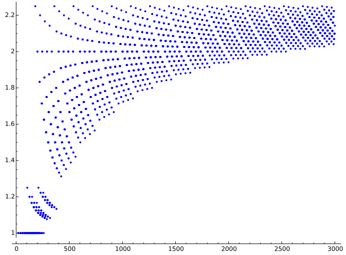
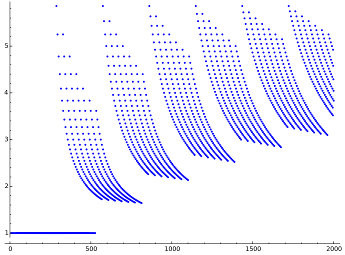


$S = \langle 20, 21, 45 \rangle$



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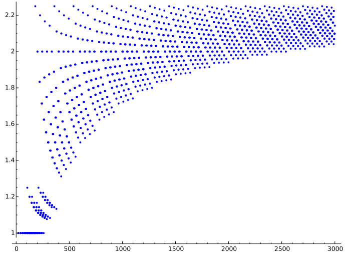

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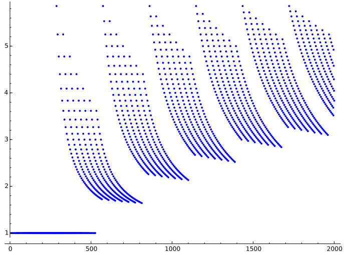

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Theorem (Barron, O., Pelayo)

Fix a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$. *For* $n \gg 0$,

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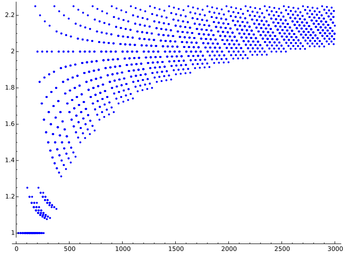

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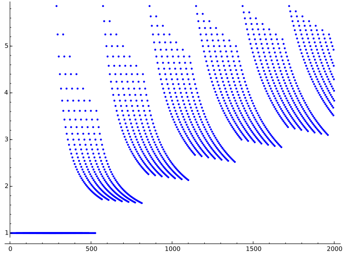

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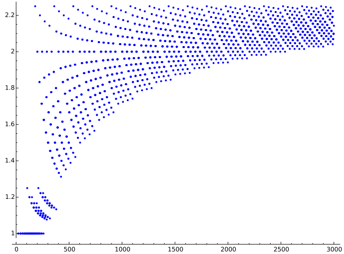

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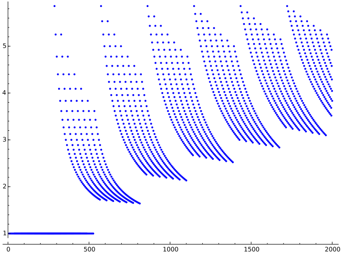

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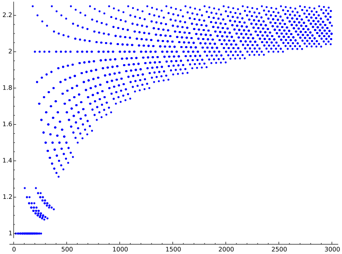
$R(S)$ is a much weaker invariant than $\mathcal{L}(S)$.



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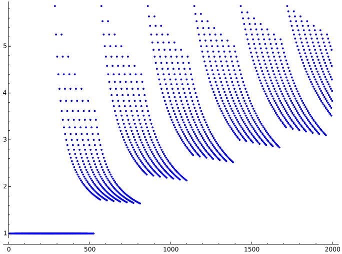

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$R(S)$ is a *much* weaker invariant than $\mathcal{L}(S)$.

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Fix arithmetical numerical monoids

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For all $n \in S$,

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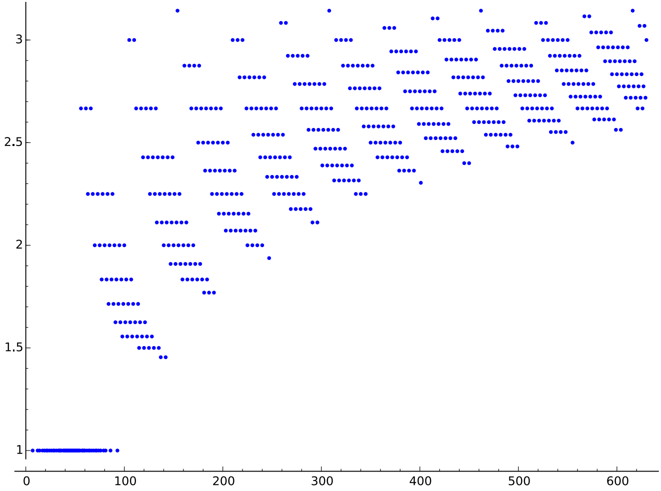
$S = \langle 7, 10, 13, 16, 19, 22 \rangle \qquad a = 7, d = 3, k = 5$



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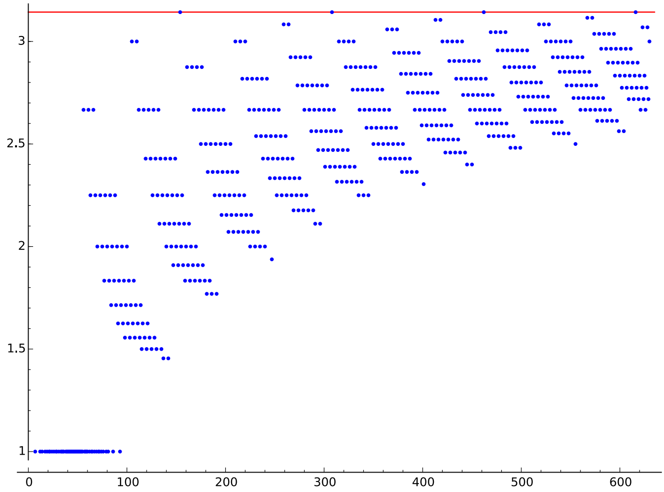

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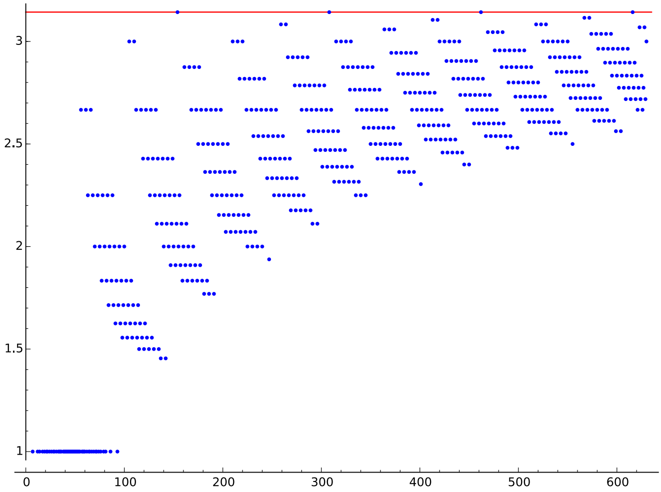

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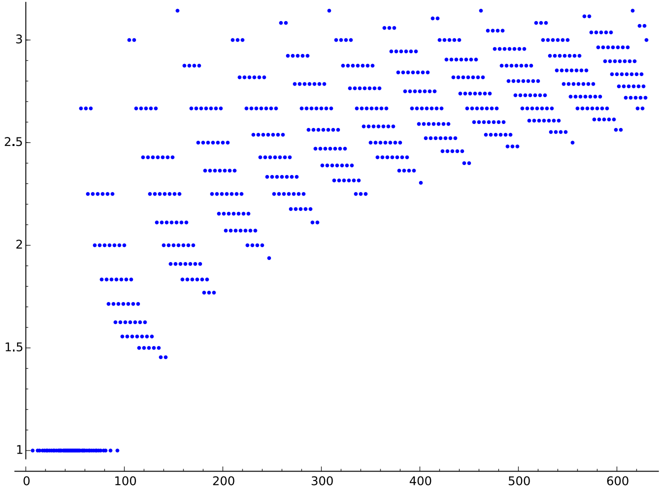

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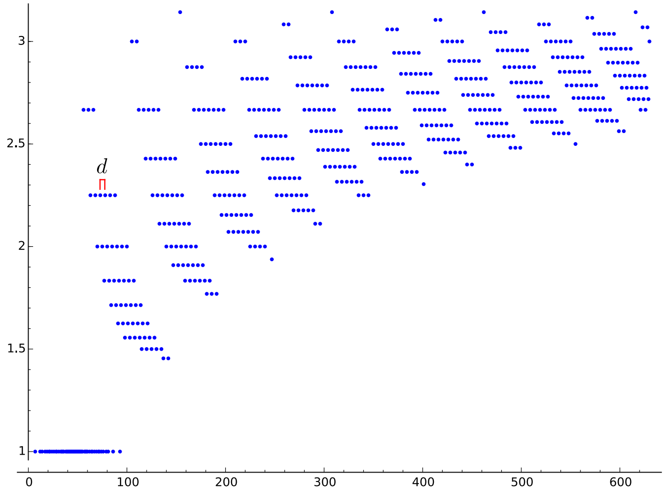

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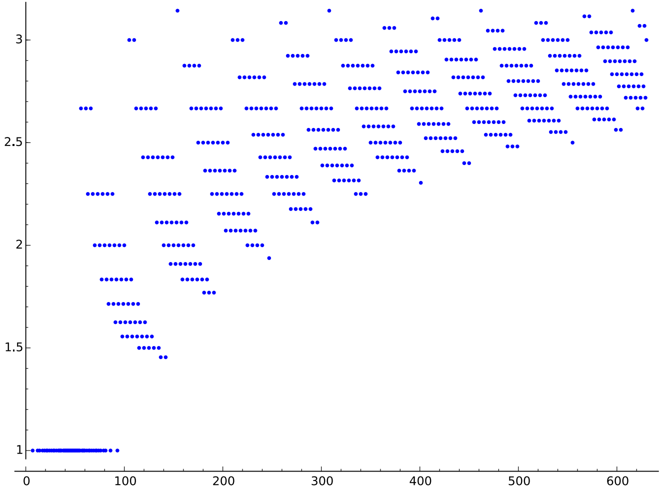

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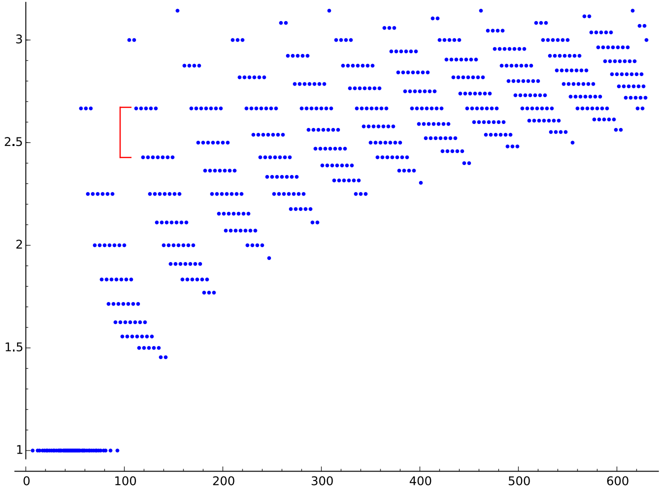

$$\frac{\alpha}{\beta} \rightsquigarrow \frac{\alpha+1}{\beta+1}$$

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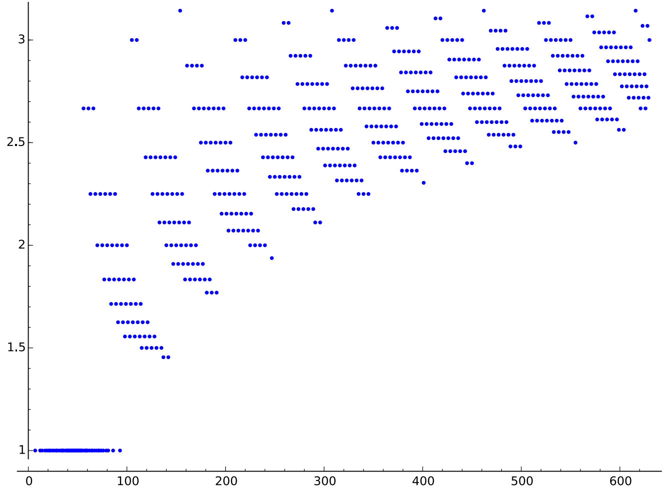

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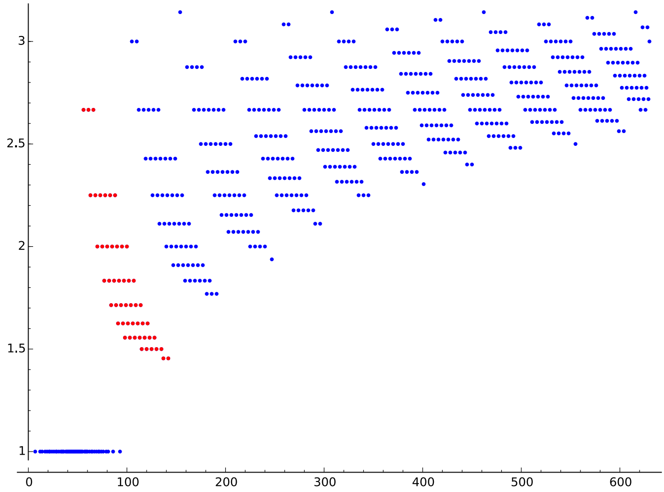

$\frac{\beta + d}{\beta}$

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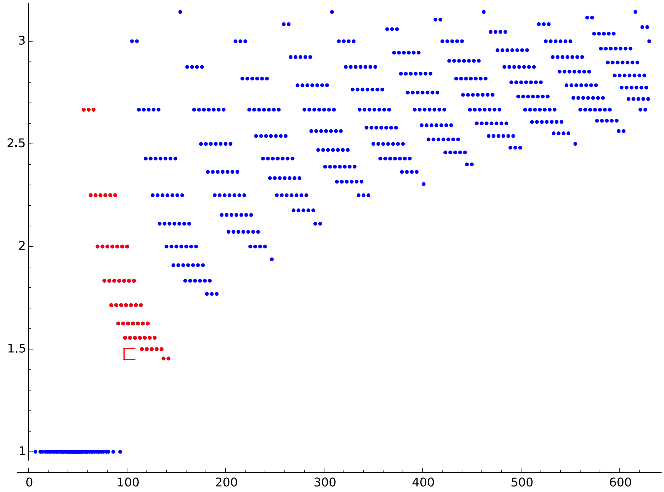

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Back to arithmetical numerical monoids

Let $S = \langle a, a+d, \ldots, a+kd \rangle$.

Theorem

If $1 < f < g$ are the smallest elements of $R(S)$, then

$$d = \frac{(g-1)(f-1)}{g-f}$$

Additionally,

$$\max R(S) = \frac{a+kd}{a} = \rho(a(a+kd))$$

What's left?

$\gcd(a, a+kd)$ determines S.

$$\langle 7, 10, 13 \rangle$$

$$\langle 14, 17, 20, 23, 26 \rangle \quad \langle 28, 31, 34, 37, 40, 43, 46, 49, 52 \rangle$$

Back to arithmetical numerical monoids

$$\langle 7, 10, 13 \rangle$$

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Back to arithmetical numerical monoids

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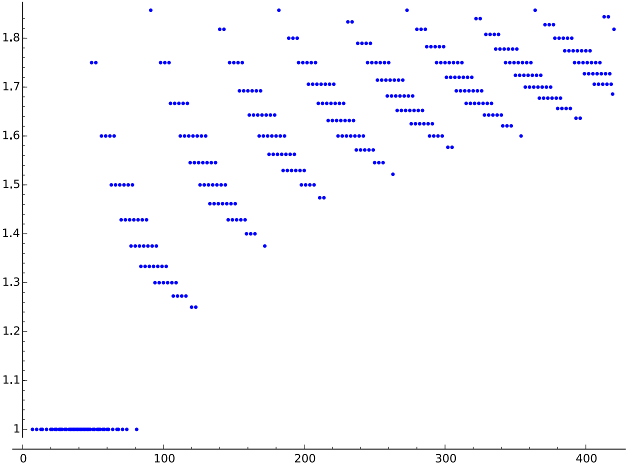

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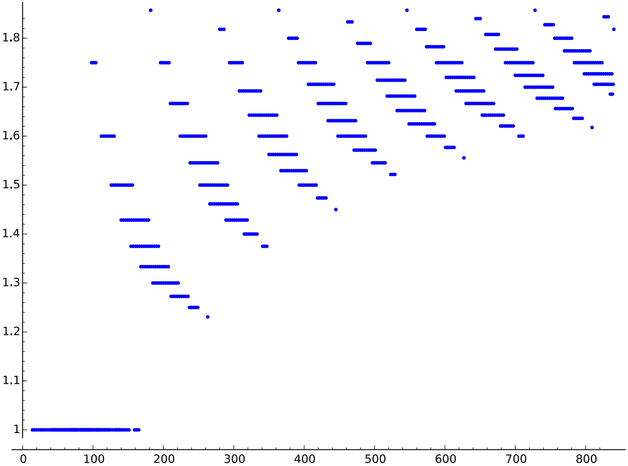

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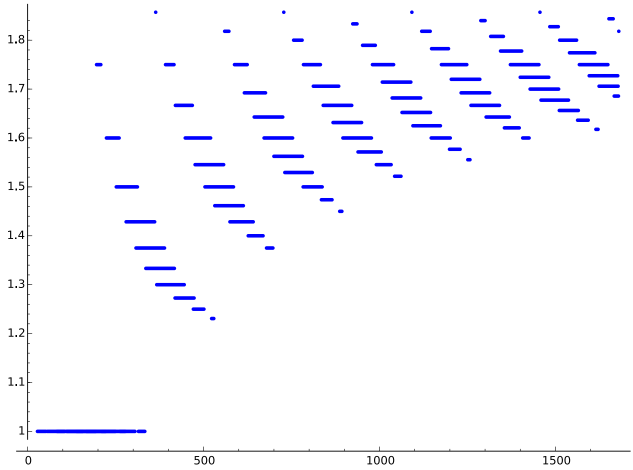

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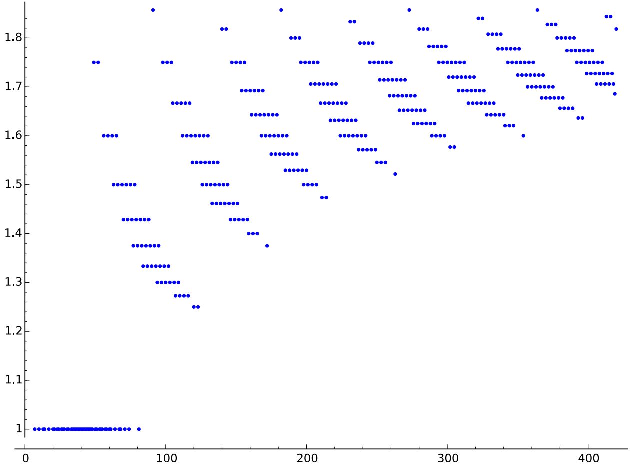

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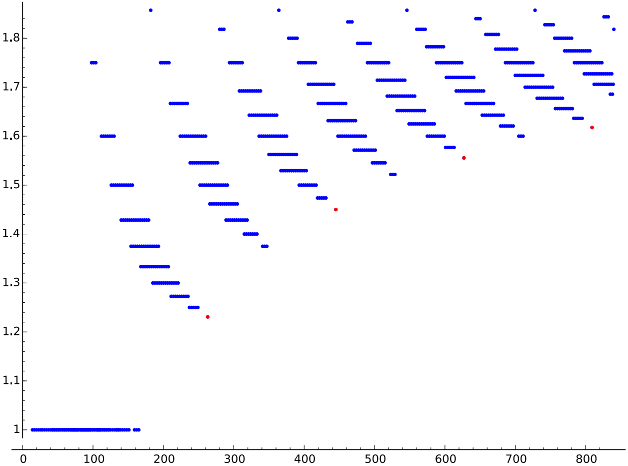

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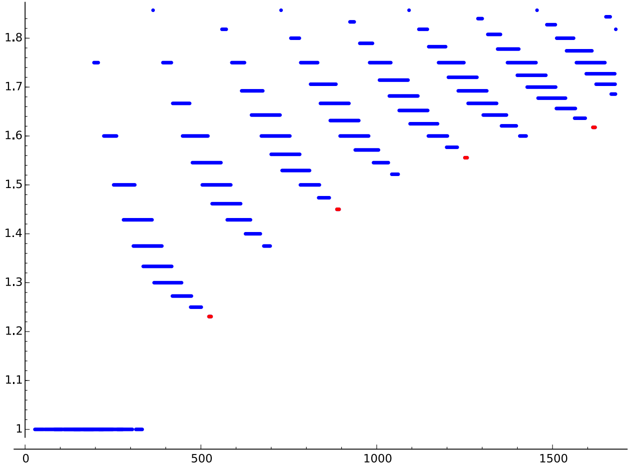

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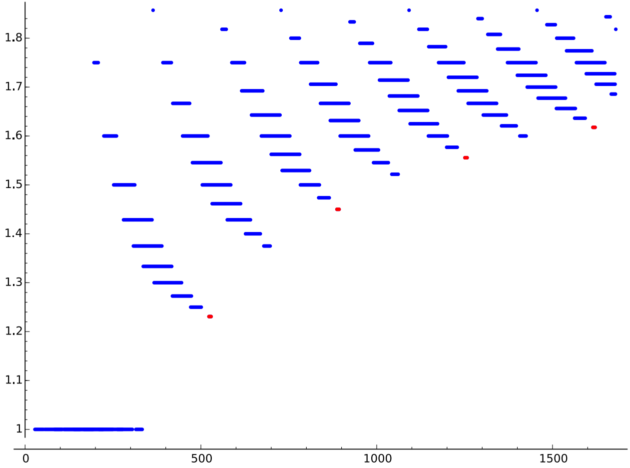
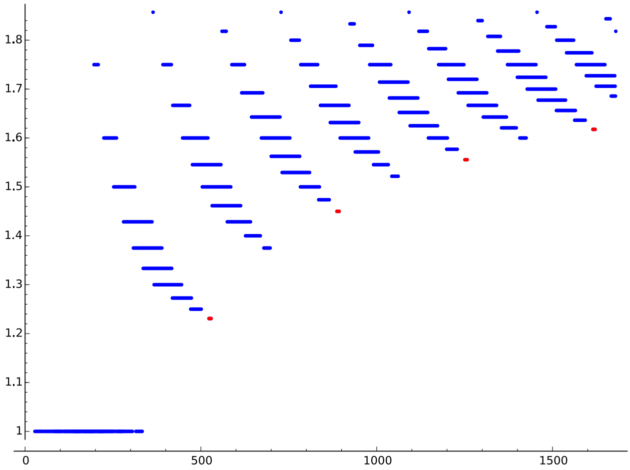


Back to arithmetical numerical monoids

Either $\gcd(a, a + kd) = 1$ or $\gcd(a, a + kd) > 1$



A surprising result

Theorem

Two distinct arithmetical numerical monoids

$$\begin{aligned} S &= \langle a, a+d, \ldots, a+kd \rangle, \\ S' &= \langle a', a'+d', \ldots, a'+k'd' \rangle \end{aligned}$$

satisfy $R(S) = R(S')$ *if and only if the all of the following hold:*

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satisfy $R(S) = R(S')$ *if and only if the all of the following hold:*

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2. $\frac{a+kd}{a} = \frac{a'+k'd'}{a'}$

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satisfy $R(S) = R(S')$ if and only if the all of the following hold:

1. $d = d'$
2. $\frac{a+kd}{a} = \frac{a'+k'd'}{a'}$
3. $\gcd(a, a+kd) > 1$ *and* $\gcd(a', a'+k'd') > 1$.

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Do these conditions look familiar?

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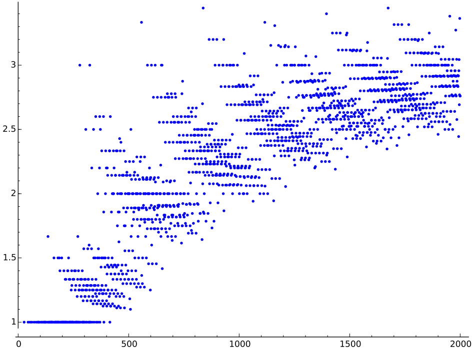
3. $\gcd(a, a+kd) > 1$ *and* $\gcd(a', a'+k'd') > 1$.

Do these conditions look familiar?

Corollary

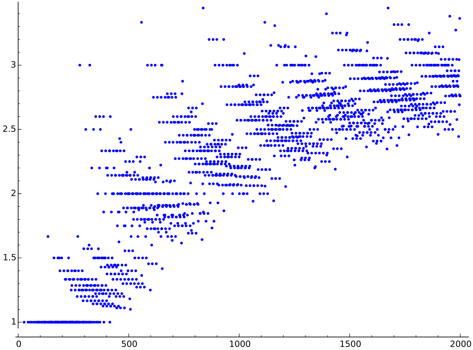
$R(S) = R(S')$ if and only if $\mathcal{L}(S) = \mathcal{L}(S')$.

The general picture



$$S = \langle 27, 45, 62, 93 \rangle$$

The general picture



$$S = \langle 27, 45, 62, 93 \rangle$$

Conjecture

If S and S' are each minimally generated by 3 elements, then $\mathcal{L}(S) = \mathcal{L}(S')$ if and only if $R(S) = R(S')$.

References

J. Amos, S. Chapman, N. Hine, J. Paixão (2007)
Sets of lengths do not characterize numerical monoids.
Integers 7 (2007) #A50.

Manuel Delgado, Pedro García-Sánchez, Jose Morais
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http://www.gap-system.org/Packages/numericalsgps.html.

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Thanks!

A curious example

Example

A simple computation shows that

$$\begin{aligned} S &= \langle 6, 10, 13, 14 \rangle, \\ S' &= \langle 6, 11, 13, 14 \rangle \end{aligned}$$

satisfy $R(S) = R(S')$ and $\{4, 6\} \in \mathcal{L}(S) \setminus \mathcal{L}(S')$.