Catenary degrees of elements in numerical monoids

Christopher O'Neill

Texas A&M University

coneill@math.tamu.edu

Joint with Vadim Ponomarenko, Reuben Tate*, and Gautam Webb*

January 11, 2015

Factorial domains

Definition

An integral domain R is *factorial* if for each non-unit $r \in R$,

1. there is a *factorization* $r = u_1 \cdots u_k$ as a product of irreducibles, and
2. this factorization is unique (up to reordering and unit multiple).

Factorial domains

Definition

An integral domain R is *factorial* if for each non-unit $r \in R$,

1. there is a *factorization* $r = u_1 \cdots u_k$ as a product of irreducibles, and
2. this factorization is unique (up to reordering and unit multiple).

Example

$\mathbb{Z}$ is factorial: each $z = p_1 \cdots p_k$ for primes $p_1 \cdots p_k$.

Atomic domains

Definition

An integral domain R is *atomic* if for each non-unit $r \in R$,

1. there is a *factorization* $r = u_1 \cdots u_k$ as a product of irreducibles, ~~and~~
2. ~~this factorization is unique (up to reordering and unit multiple~~).

Example

$\mathbb{Z}$ is factorial: each $z = p_1 \cdots p_k$ for primes $p_1 \cdots p_k$.

Atomic domains

Definition

An integral domain R is *atomic* if for each non-unit $r \in R$,

1. there is a *factorization* $r = u_1 \cdots u_k$ as a product of irreducibles, ~~and~~
2. ~~this factorization is unique (up to reordering and unit multiple)~~.

Example

$\mathbb{Z}$ is factorial: each $z = p_1 \cdots p_k$ for primes $p_1 \cdots p_k$.

Example

If $R = \mathbb{Z}[\sqrt{-5}]$, then $6 \in R$ has two distinct factorizations:

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

Atomic domains

Definition

An integral domain R is *atomic* if for each non-unit $r \in R$,

1. there is a *factorization* $r = u_1 \cdots u_k$ as a product of irreducibles, ~~and~~
2. ~~this factorization is unique (up to reordering and unit multiple)~~.

Example

$\mathbb{Z}$ is factorial: each $z = p_1 \cdots p_k$ for primes $p_1 \cdots p_k$.

Example

If $R = \mathbb{Z}[\sqrt{-5}]$, then $6 \in R$ has two distinct factorizations:

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

To prove: define a *valuation* $a + b\sqrt{-5} \mapsto a^2 + 5b^2$.

Atomic domains

Definition

An integral domain R is *atomic* if for each non-unit $r \in R$,

1. there is a *factorization* $r = u_1 \cdots u_k$ as a product of irreducibles, ~~and~~
2. ~~this factorization is unique (up to reordering and unit multiple)~~.

Example

$\mathbb{Z}$ is factorial: each $z = p_1 \cdots p_k$ for primes $p_1 \cdots p_k$.

Example

If $R = \mathbb{Z}[\sqrt{-5}]$, then $6 \in R$ has two distinct factorizations:

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

To prove: define a *valuation* $a + b\sqrt{-5} \mapsto a^2 + 5b^2$.
The point: it's involved.

Atomic domains

Definition

An integral domain R is *atomic* if for each non-unit $r \in R$,

1. there is a *factorization* $r = u_1 \cdots u_k$ as a product of irreducibles, ~~and~~
2. ~~this factorization is unique (up to reordering and unit multiple)~~.

Example

$\mathbb{Z}$ is factorial: each $z = p_1 \cdots p_k$ for primes $p_1 \cdots p_k$.

Example

Atomic domains

Definition

An integral domain R is *atomic* if for each non-unit $r \in R$,

1. there is a *factorization* $r = u_1 \cdots u_k$ as a product of irreducibles, ~~and~~
2. ~~this factorization is unique (up to reordering and unit multiple~~).

Example

$\mathbb{Z}$ is factorial: each $z = p_1 \cdots p_k$ for primes $p_1 \cdots p_k$.

Example

Let $R = \mathbb{C}[x^2, x^3]$.

Atomic domains

Definition

An integral domain R is *atomic* if for each non-unit $r \in R$,

1. there is a *factorization* $r = u_1 \cdots u_k$ as a product of irreducibles, ~~and~~
2. ~~this factorization is unique (up to reordering and unit multiple~~).

Example

$\mathbb{Z}$ is factorial: each $z = p_1 \cdots p_k$ for primes $p_1 \cdots p_k$.

Example

Let $R = \mathbb{C}[x^2, x^3]$.

1. x^2 and x^3 are irreducible.

Atomic domains

Definition

An integral domain R is *atomic* if for each non-unit $r \in R$,

1. there is a *factorization* $r = u_1 \cdots u_k$ as a product of irreducibles, ~~and~~
2. ~~this factorization is unique (up to reordering and unit multiple~~).

Example

$\mathbb{Z}$ is factorial: each $z = p_1 \cdots p_k$ for primes $p_1 \cdots p_k$.

Example

Let $R = \mathbb{C}[x^2, x^3]$.

1. x^2 and x^3 are irreducible.
2. $x^6 = x^3 \cdot x^3 = x^2 \cdot x^2 \cdot x^2$.

Atomic domains

Definition

An integral domain R is *atomic* if for each non-unit $r \in R$,

1. there is a *factorization* $r = u_1 \cdots u_k$ as a product of irreducibles, ~~and~~
2. ~~this factorization is unique (up to reordering and unit multiple)~~.

Example

$\mathbb{Z}$ is factorial: each $z = p_1 \cdots p_k$ for primes $p_1 \cdots p_k$.

Observation

Atomic domains

Definition

An integral domain R is *atomic* if for each non-unit $r \in R$,

1. there is a *factorization* $r = u_1 \cdots u_k$ as a product of irreducibles, ~~and~~
2. ~~this factorization is unique (up to reordering and unit multiple)~~.

Example

$\mathbb{Z}$ is factorial: each $z = p_1 \cdots p_k$ for primes $p_1 \cdots p_k$.

Observation

- Where's the addition?

Atomic domains

Definition

An integral domain R is *atomic* if for each non-unit $r \in R$,

1. there is a *factorization* $r = u_1 \cdots u_k$ as a product of irreducibles, ~~and~~
2. ~~this factorization is unique (up to reordering and unit multiple)~~.

Example

$\mathbb{Z}$ is factorial: each $z = p_1 \cdots p_k$ for primes $p_1 \cdots p_k$.

Observation

- Where's the addition?
- Factorization in (cancellative comutative) monoids:

Atomic domains

Definition

An integral domain R is *atomic* if for each non-unit $r \in R$,

1. there is a *factorization* $r = u_1 \cdots u_k$ as a product of irreducibles, ~~and~~
2. ~~this factorization is unique (up to reordering and unit multiple)~~.

Example

$\mathbb{Z}$ is factorial: each $z = p_1 \cdots p_k$ for primes $p_1 \cdots p_k$.

Observation

- Where's the addition?
- Factorization in (cancellative comutative) monoids:

$$(R, +, \cdot) \quad \rightsquigarrow \quad (R \setminus \{0\}, \cdot)$$

Atomic domains

Definition

An integral domain R is *atomic* if for each non-unit $r \in R$,

1. there is a *factorization* $r = u_1 \cdots u_k$ as a product of irreducibles, ~~and~~
2. ~~this factorization is unique (up to reordering and unit multiple)~~.

Example

$\mathbb{Z}$ is factorial: each $z = p_1 \cdots p_k$ for primes $p_1 \cdots p_k$.

Observation

- Where's the addition?
- Factorization in (cancellative comutative) monoids:

$$\begin{array}{lcl} (R, +, \cdot) & \rightsquigarrow & (R \setminus \{0\}, \cdot) \\ (\mathbb{C}[M], +, \cdot) & \rightsquigarrow & (M, \cdot) \end{array}$$

Numerical monoids

Definition

A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

Numerical monoids

Definition

A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

Example

Let $S = \langle 2, 3 \rangle = \{2, 3, 4, 5, \ldots\}$ under **addition**.

Numerical monoids

Definition

A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

Example

Let $S = \langle 2, 3 \rangle = \{2, 3, 4, 5, \ldots\}$ under **addition**. $\mathbb{C}[S] = \mathbb{C}[x^2, x^3]$.

Numerical monoids

Definition

A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

Example

Let $S = \langle 2, 3 \rangle = \{2, 3, 4, 5, \ldots\}$ under **addition**. $\mathbb{C}[S] = \mathbb{C}[x^2, x^3]$.

$$x^6 = x^3 \cdot x^3 = x^2 \cdot x^2 \cdot x^2$$

Numerical monoids

Definition

A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

Example

Let $S = \langle 2, 3 \rangle = \{2, 3, 4, 5, \ldots\}$ under **addition**. $\mathbb{C}[S] = \mathbb{C}[x^2, x^3]$.

$$x^6 = x^3 \cdot x^3 = x^2 \cdot x^2 \cdot x^2 \quad \rightsquigarrow \quad 6 = 3 + 3 = 2 + 2 + 2.$$

Numerical monoids

Definition

A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

Example

Let $S = \langle 2, 3 \rangle = \{2, 3, 4, 5, \ldots\}$ under **addition**. $\mathbb{C}[S] = \mathbb{C}[x^2, x^3]$.

$$x^6 = x^3 \cdot x^3 = x^2 \cdot x^2 \cdot x^2 \quad \rightsquigarrow \quad 6 = 3 + 3 = 2 + 2 + 2.$$

Factorizations are additive!

Numerical monoids

Definition

A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

Example

Let $S = \langle 2, 3 \rangle = \{2, 3, 4, 5, \ldots\}$ under **addition**. $\mathbb{C}[S] = \mathbb{C}[x^2, x^3]$.

$$x^6 = x^3 \cdot x^3 = x^2 \cdot x^2 \cdot x^2 \quad \rightsquigarrow \quad 6 = 3 + 3 = 2 + 2 + 2.$$

Factorizations are additive!

Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$.

Numerical monoids

Definition

A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

Example

Let $S = \langle 2, 3 \rangle = \{2, 3, 4, 5, \ldots\}$ under **addition**. $\mathbb{C}[S] = \mathbb{C}[x^2, x^3]$.

$$x^6 = x^3 \cdot x^3 = x^2 \cdot x^2 \cdot x^2 \quad \rightsquigarrow \quad 6 = 3 + 3 = 2 + 2 + 2.$$

Factorizations are additive!

Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$. "McNugget Monoid"

Numerical monoids

Definition

A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

Example

Let $S = \langle 2, 3 \rangle = \{2, 3, 4, 5, \ldots\}$ under **addition**. $\mathbb{C}[S] = \mathbb{C}[x^2, x^3]$.

$$x^6 = x^3 \cdot x^3 = x^2 \cdot x^2 \cdot x^2 \quad \rightsquigarrow \quad 6 = 3 + 3 = 2 + 2 + 2.$$

Factorizations are additive!

Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$. "McNugget Monoid"

$$60 \quad = \quad 7(6) + 2(9)$$

Numerical monoids

Definition

A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

Example

Let $S = \langle 2, 3 \rangle = \{2, 3, 4, 5, \ldots\}$ under **addition**. $\mathbb{C}[S] = \mathbb{C}[x^2, x^3]$.

$$x^6 = x^3 \cdot x^3 = x^2 \cdot x^2 \cdot x^2 \quad \rightsquigarrow \quad 6 = 3 + 3 = 2 + 2 + 2.$$

Factorizations are additive!

Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$. "McNugget Monoid"

$$\begin{array}{rcl} 60 & = & 7(6) + 2(9) \\ & = & \qquad\qquad 3(20) \end{array}$$

Numerical monoids

Definition

A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

Example

Let $S = \langle 2, 3 \rangle = \{2, 3, 4, 5, \ldots\}$ under **addition**. $\mathbb{C}[S] = \mathbb{C}[x^2, x^3]$.

$$x^6 = x^3 \cdot x^3 = x^2 \cdot x^2 \cdot x^2 \quad \rightsquigarrow \quad 6 = 3 + 3 = 2 + 2 + 2.$$

Factorizations are additive!

Example

$McN = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$. "McNugget Monoid"

$$\begin{array}{rclcc} 60 & = & 7(6) + 2(9) & \rightsquigarrow & (7, 2, 0) \\ & = & 3(20) & \rightsquigarrow & (0, 0, 3) \end{array}$$

Factorization invariants: towards the catenary degree

Factorization invariants: towards the catenary degree

Definition

Fix a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$,

$$\mathsf{Z}_S(n) = \left\{(a_1, \ldots, a_k) \in \mathbb{N}^k : n = a_1 n_1 + \cdots + a_k n_k\right\}$$

denotes the *set of factorizations* of n.

Factorization invariants: towards the catenary degree

Definition

Fix a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$,

$$\mathsf{Z}_S(n) = \left\{(a_1, \ldots, a_k) \in \mathbb{N}^k : n = a_1 n_1 + \cdots + a_k n_k\right\}$$

denotes the *set of factorizations* of n. Equivalently, if

$$\begin{array}{rcl} \phi : \mathbb{N}^k & \longrightarrow & S \\ \vec{e}_i & \longmapsto & n_i \end{array}$$

then $\mathsf{Z}_S(n) = \phi^{-1}(n)$.

Factorization invariants: towards the catenary degree

Definition

Fix a numerical monoid $S = \langle n_1, \dots, n_k \rangle$. For $n \in S$,

$$\mathsf{Z}_S(n) = \left\{(a_1, \dots, a_k) \in \mathbb{N}^k : n = a_1 n_1 + \cdots + a_k n_k\right\}$$

denotes the *set of factorizations* of n. Equivalently, if

$$\begin{array}{rcl} \phi : \mathbb{N}^k & \longrightarrow & S \\ \vec{e}_i & \longmapsto & n_i \end{array}$$

then $\mathsf{Z}_S(n) = \phi^{-1}(n)$. For $f, f' \in \mathsf{Z}_S(n)$,

$$|f| \quad = \quad f_1 + \cdots + f_k$$

Factorization invariants: towards the catenary degree

Definition

Fix a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$,

$$\mathsf{Z}_S(n) = \left\{(a_1, \ldots, a_k) \in \mathbb{N}^k : n = a_1 n_1 + \cdots + a_k n_k\right\}$$

denotes the *set of factorizations* of n. Equivalently, if

$$\begin{array}{rcl} \phi : \mathbb{N}^k & \longrightarrow & S \\ \vec{e}_i & \longmapsto & n_i \end{array}$$

then $\mathsf{Z}_S(n) = \phi^{-1}(n)$. For $f, f' \in \mathsf{Z}_S(n)$,

$$|f| \quad = \quad f_1 + \cdots + f_k \qquad (\textit{length} \text{ of } f)$$

Factorization invariants: towards the catenary degree

Definition

Fix a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$,

$$\mathsf{Z}_S(n) = \left\{(a_1, \ldots, a_k) \in \mathbb{N}^k : n = a_1 n_1 + \cdots + a_k n_k\right\}$$

denotes the *set of factorizations* of n. Equivalently, if

$$\begin{array}{rcl} \phi : \mathbb{N}^k & \longrightarrow & S \\ \vec{e}_i & \longmapsto & n_i \end{array}$$

then $\mathsf{Z}_S(n) = \phi^{-1}(n)$. For $f, f' \in \mathsf{Z}_S(n)$,

$$\begin{array}{rcl} |f| & = & f_1 + \cdots + f_k \quad (\textit{length} \text{ of } f) \\ \gcd(f, f') & = & (\min(f_1, f_1'), \ldots, \min(f_k, f_k')) \end{array}$$

Factorization invariants: towards the catenary degree

Definition

Fix a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$,

$$\mathsf{Z}_S(n) = \left\{(a_1, \ldots, a_k) \in \mathbb{N}^k : n = a_1 n_1 + \cdots + a_k n_k\right\}$$

denotes the *set of factorizations* of n. Equivalently, if

$$\begin{array}{rcl} \phi : \mathbb{N}^k & \longrightarrow & S \\ \vec{e}_i & \longmapsto & n_i \end{array}$$

then $\mathsf{Z}_S(n) = \phi^{-1}(n)$. For $f, f' \in \mathsf{Z}_S(n)$,

$$\begin{array}{rcl} |f| & = & f_1 + \cdots + f_k \quad (\textit{length} \text{ of } f) \\ \gcd(f, f') & = & (\min(f_1, f_1'), \ldots, \min(f_k, f_k')) \\ d(f, f') & = & \max\{|f - \gcd(f, f')|, |f' - \gcd(f, f')|\} \end{array}$$

Factorization invariants: towards the catenary degree

Example

$S = \langle 4, 6, 7 \rangle \subset \mathbb{N}$,

Factorization invariants: towards the catenary degree

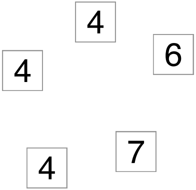
Example

$S = \langle 4, 6, 7 \rangle \subset \mathbb{N}$, $f = (3, 1, 1), f' = (1, 0, 3) \in \mathsf{Z}_S(25)$.

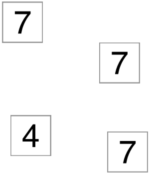
Factorization invariants: towards the catenary degree

Example

$S = \langle 4, 6, 7 \rangle \subset \mathbb{N}$, $f = (3, 1, 1)$, $f' = (1, 0, 3) \in \mathsf{Z}_S(25)$.



(3,1,1)

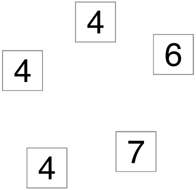


(1,0,3)

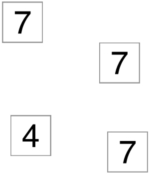
Example

$S = \langle 4, 6, 7 \rangle \subset \mathbb{N}$, $f = (3, 1, 1)$, $f' = (1, 0, 3) \in Z_S(25)$.

- $g = \gcd(f, f')$



(3,1,1)

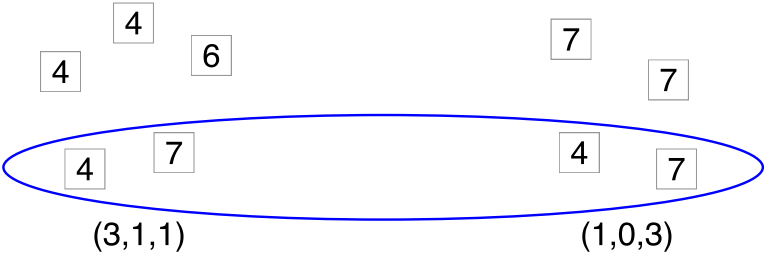


(1,0,3)

Example

$S = \langle 4, 6, 7 \rangle \subset \mathbb{N}$, $f = (3, 1, 1)$, $f' = (1, 0, 3) \in \mathsf{Z}_S(25)$.

- $g = \gcd(f, f')$

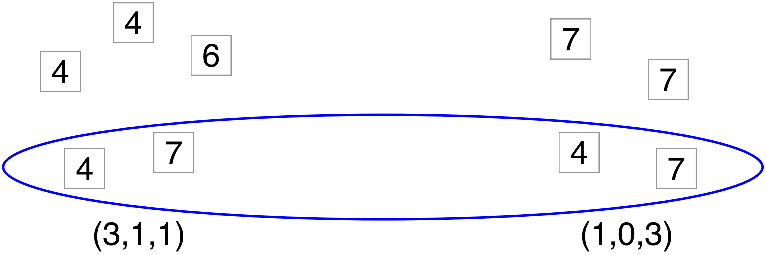


Factorization invariants: towards the catenary degree

Example

$S = \langle 4, 6, 7 \rangle \subset \mathbb{N}$, $f = (3, 1, 1)$, $f' = (1, 0, 3) \in \mathsf{Z}_S(25)$.

- $g = \gcd(f, f') = (1, 0, 1)$.

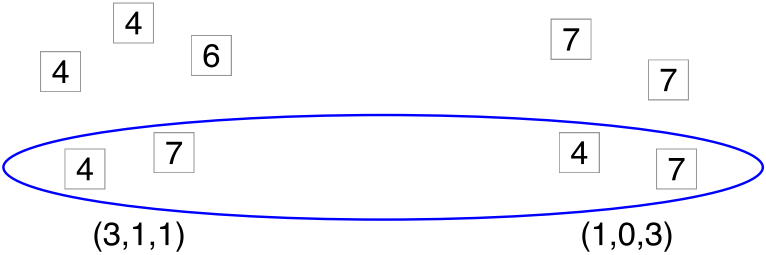


Factorization invariants: towards the catenary degree

Example

$S = \langle 4, 6, 7\rangle \subset \mathbb{N}$, $f = (3, 1, 1)$, $f' = (1, 0, 3) \in \mathsf{Z}_S(25)$.

- $g = \gcd(f, f') = (1, 0, 1)$.
- $d(f, f')$

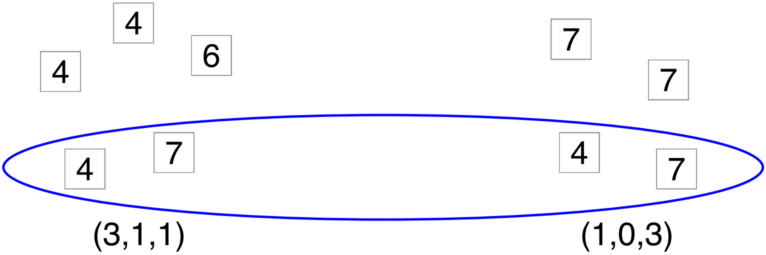


Factorization invariants: towards the catenary degree

Example

$S = \langle 4, 6, 7 \rangle \subset \mathbb{N}$, $f = (3, 1, 1), f' = (1, 0, 3) \in Z_S(25)$.

- $g = \gcd(f, f') = (1, 0, 1)$.
- $d(f, f') = \max\{|f - g|, |f' - g|\}$

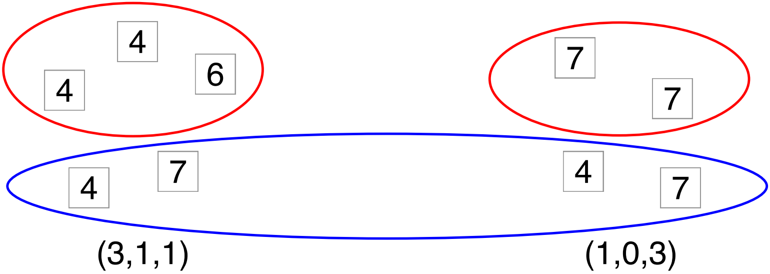


Factorization invariants: towards the catenary degree

Example

$S = \langle 4, 6, 7 \rangle \subset \mathbb{N}$, $f = (3, 1, 1), f' = (1, 0, 3) \in Z_S(25)$.

- $g = \gcd(f, f') = (1, 0, 1)$.
- $d(f, f') = \max\{|f - g|, |f' - g|\}$

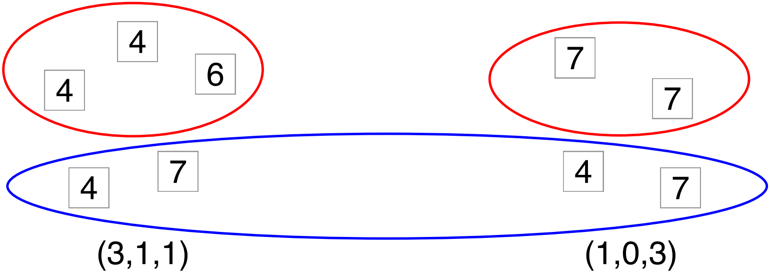


Factorization invariants: towards the catenary degree

Example

$S = \langle 4, 6, 7 \rangle \subset \mathbb{N}$, $f = (3, 1, 1)$, $f' = (1, 0, 3) \in \mathsf{Z}_S(25)$.

- $g = \gcd(f, f') = (1, 0, 1)$.
- $d(f, f') = \max\{|f - g|, |f' - g|\} = 3$.



The catenary degree

Definition

Fix a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$, define the *catenary degree* $c(n)$ as follows:

The catenary degree

Definition

Fix a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$, define the *catenary degree* $c(n)$ as follows:

1. Construct a complete graph G with vertex set $\mathsf{Z}_S(n)$ where each edge (f, f') has label $d(f, f')$ (*catenary graph*).

The catenary degree

Definition

Fix a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$, define the *catenary degree* $c(n)$ as follows:

1. Construct a complete graph G with vertex set $\mathsf{Z}_S(n)$ where each edge (f, f') has label $d(f, f')$ (*catenary graph*).
2. Locate the largest edge weight e in G.

The catenary degree

Definition

Fix a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$, define the *catenary degree* $c(n)$ as follows:

1. Construct a complete graph G with vertex set $\mathsf{Z}_S(n)$ where each edge (f, f') has label $d(f, f')$ (*catenary graph*).
2. Locate the largest edge weight e in G.
3. Remove all edges from G with weight e.

The catenary degree

Definition

Fix a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$, define the *catenary degree* $c(n)$ as follows:

1. Construct a complete graph G with vertex set $Z_S(n)$ where each edge (f, f') has label $d(f, f')$ (*catenary graph*).
2. Locate the largest edge weight e in G.
3. Remove all edges from G with weight e.
4. If G is disconnected, return e. Otherwise, return to step 2.

The catenary degree

Definition

Fix a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$, define the *catenary degree* $c(n)$ as follows:

1. Construct a complete graph G with vertex set $\mathsf{Z}_S(n)$ where each edge (f, f') has label $d(f, f')$ (*catenary graph*).
2. Locate the largest edge weight e in G.
3. Remove all edges from G with weight e.
4. If G is disconnected, return e. Otherwise, return to step 2.

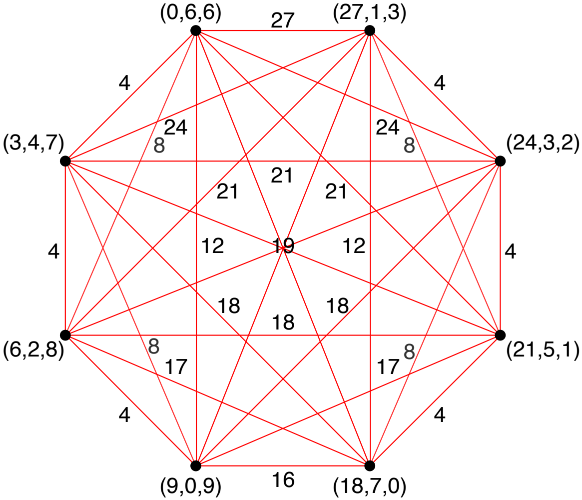
If $|\mathsf{Z}_S(n)| = 1$, define $c(n) = 0$.

A Big Example

$S = \langle 11, 36, 39 \rangle, n = 450$

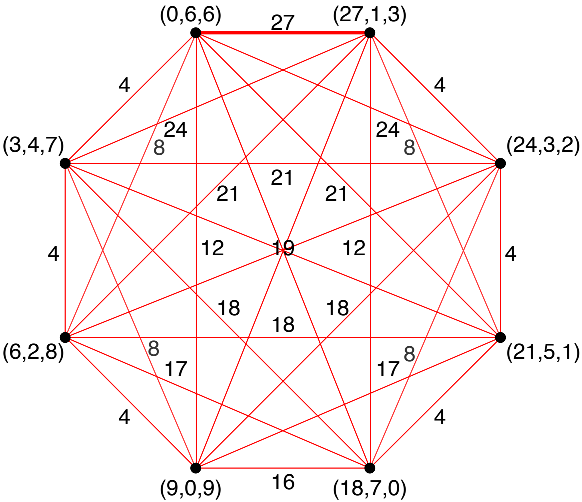
A Big Example

$S = \langle 11, 36, 39 \rangle, n = 450$



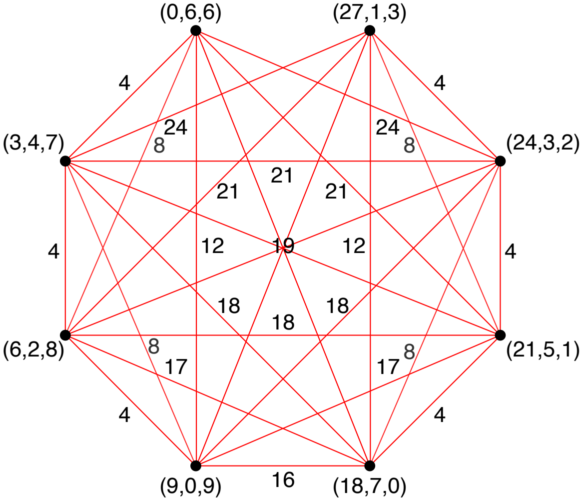
A Big Example

$S = \langle 11, 36, 39 \rangle, n = 450$



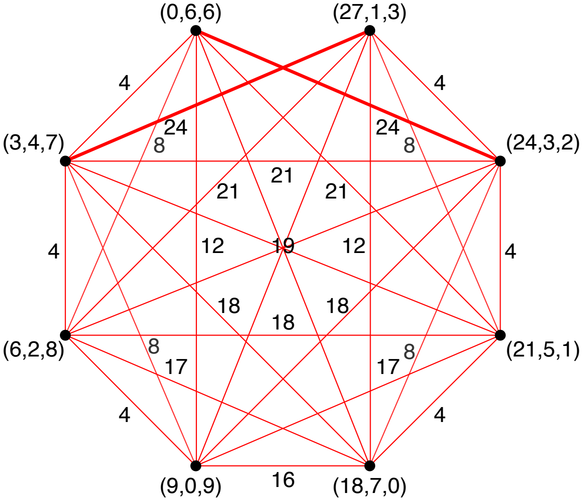
A Big Example

$S = \langle 11, 36, 39 \rangle, n = 450$



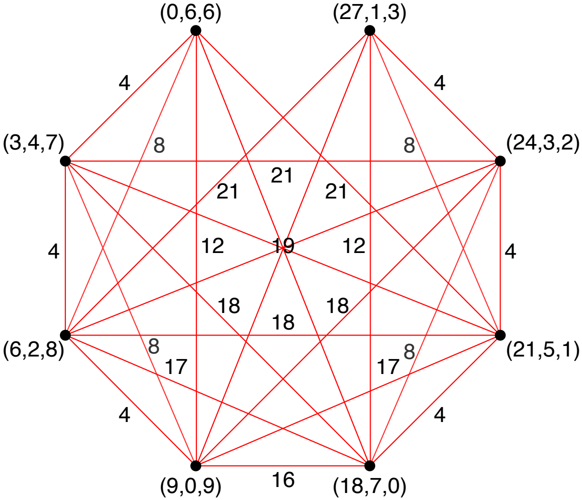
A Big Example

$S = \langle 11, 36, 39 \rangle, n = 450$



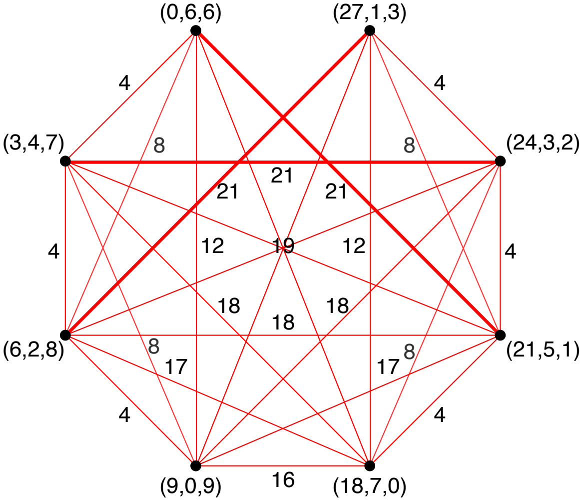
A Big Example

$S = \langle 11, 36, 39 \rangle, n = 450$



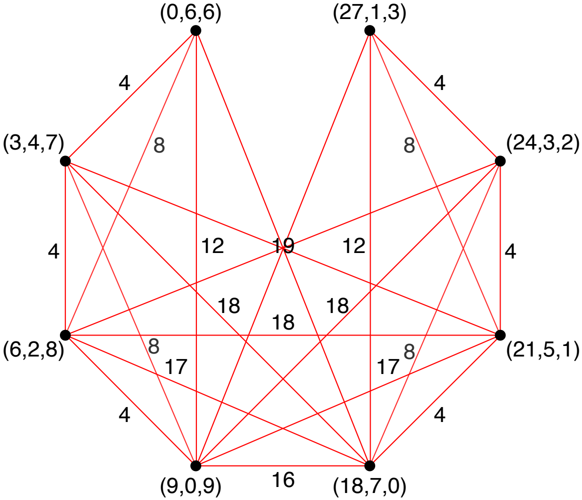
A Big Example

$S = \langle 11, 36, 39 \rangle, n = 450$



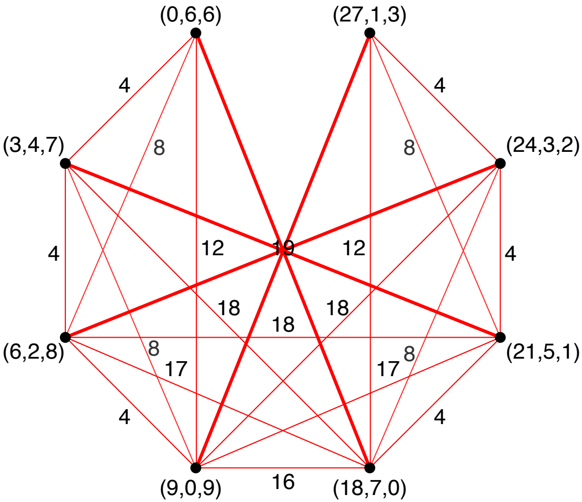
A Big Example

$S = \langle 11, 36, 39 \rangle, n = 450$



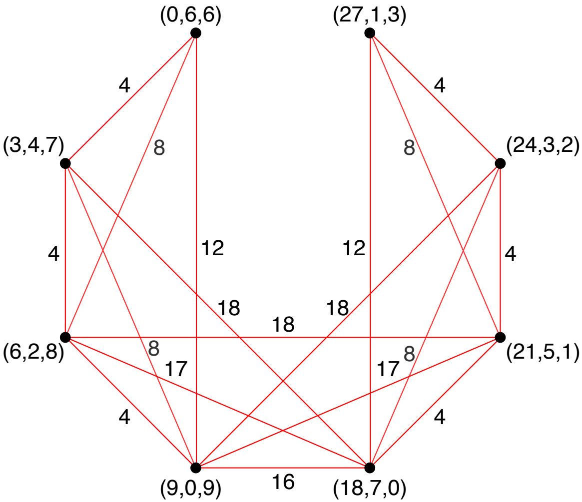
A Big Example

$S = \langle 11, 36, 39 \rangle, n = 450$



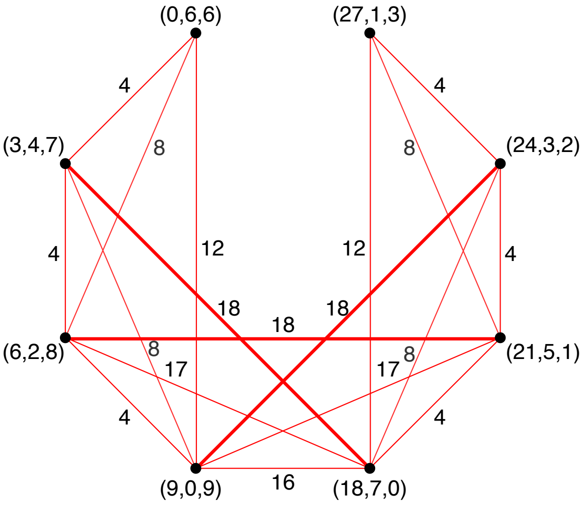
A Big Example

$S = \langle 11, 36, 39 \rangle, n = 450$



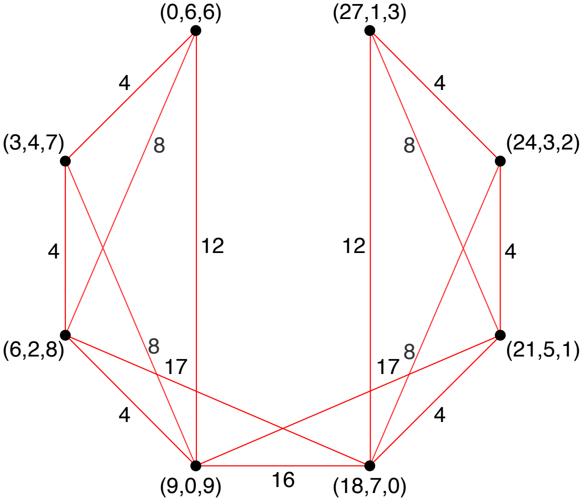
A Big Example

$S = \langle 11, 36, 39 \rangle, n = 450$



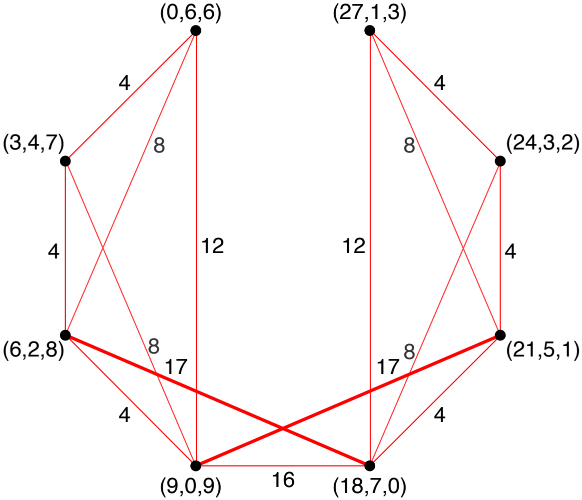
A Big Example

$S = \langle 11, 36, 39 \rangle, n = 450$



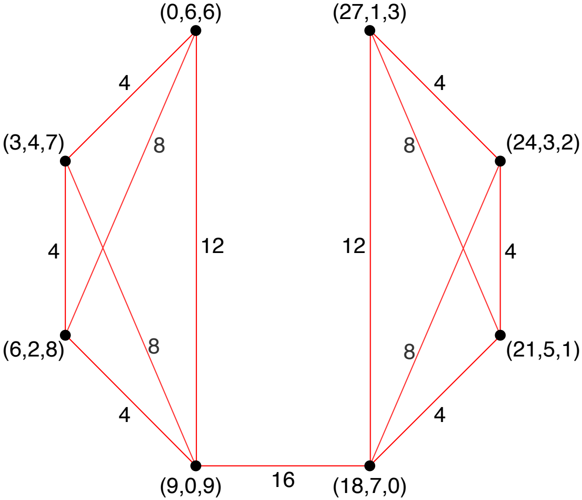
A Big Example

$S = \langle 11, 36, 39 \rangle, n = 450$



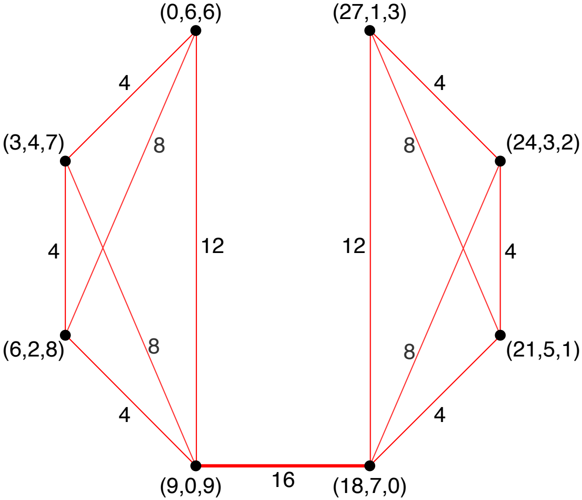
A Big Example

$S = \langle 11, 36, 39 \rangle, n = 450$



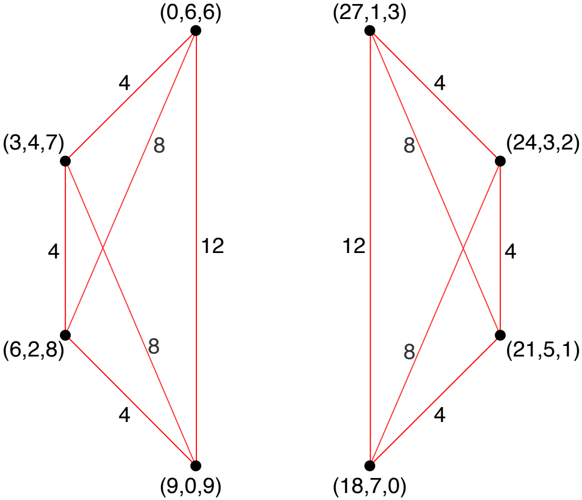
A Big Example

$S = \langle 11, 36, 39 \rangle, n = 450$



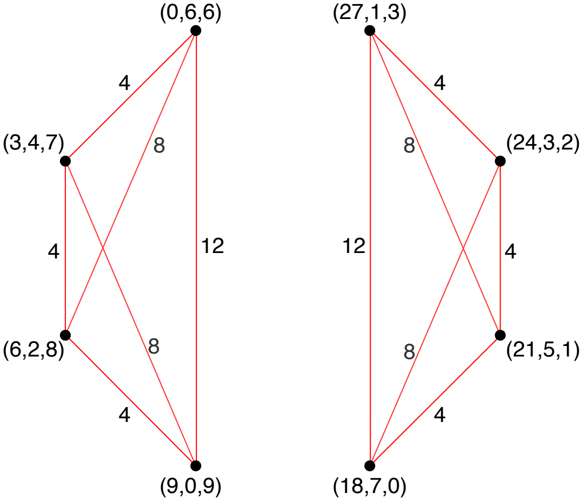
A Big Example

$S = \langle 11, 36, 39 \rangle, n = 450$



A Big Example

$S = \langle 11, 36, 39\rangle, n = 450,\ c(n) = 16$

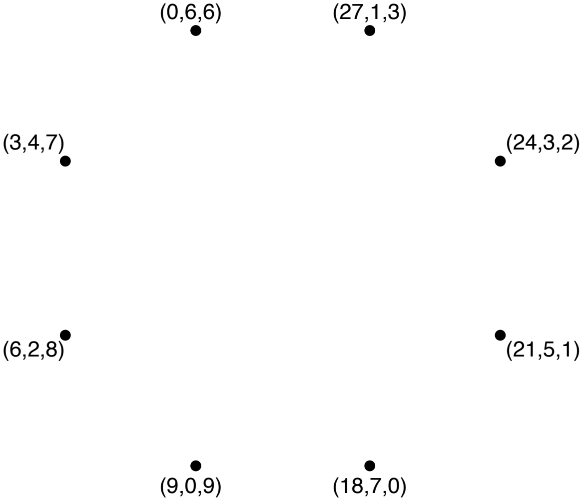


A Big Example, Method 2

$S = \langle 11, 36, 39 \rangle, n = 450$

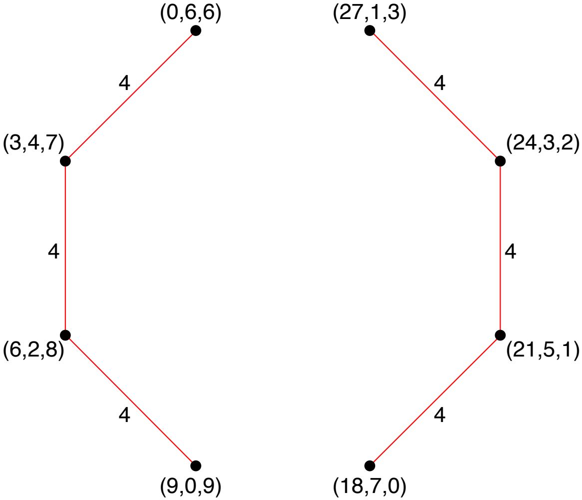
A Big Example, Method 2

$S = \langle 11, 36, 39 \rangle, n = 450$



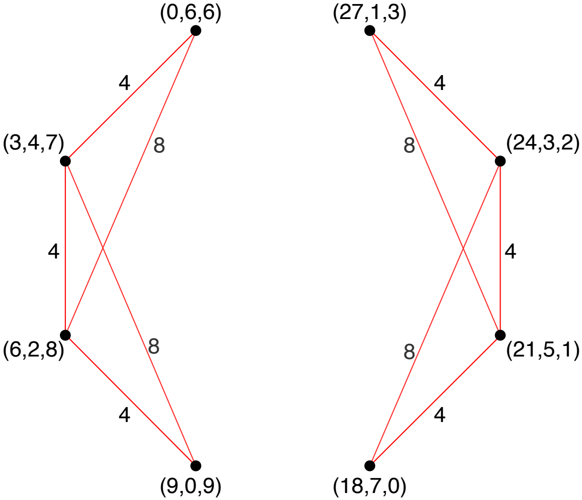
A Big Example, Method 2

$S = \langle 11, 36, 39 \rangle$, $n = 450$



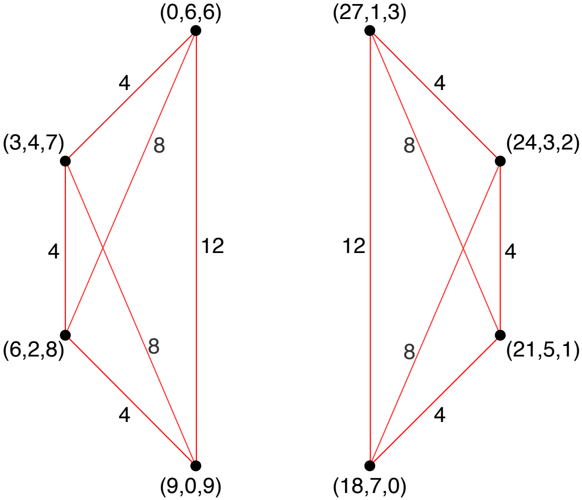
A Big Example, Method 2

$S = \langle 11, 36, 39 \rangle, n = 450$



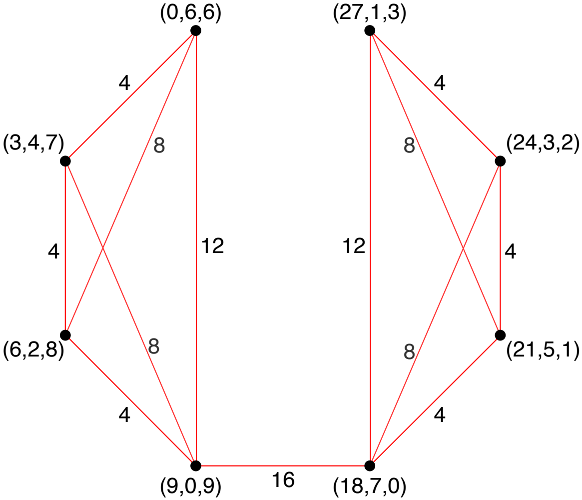
A Big Example, Method 2

$S = \langle 11, 36, 39 \rangle, n = 450$



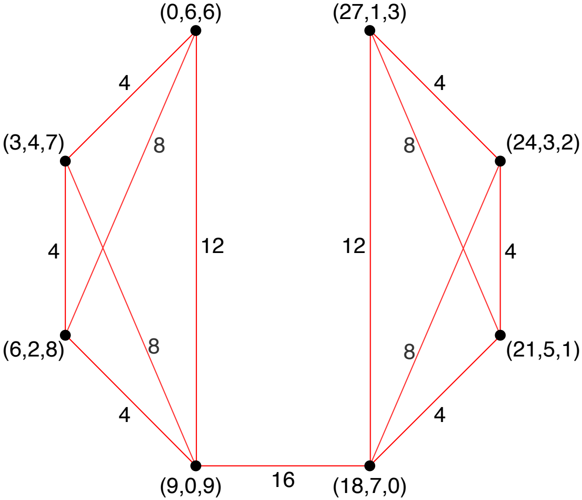
A Big Example, Method 2

$S = \langle 11, 36, 39 \rangle, n = 450$



A Big Example, Method 2

$S = \langle 11, 36, 39 \rangle, n = 450, \ c(n) = 16$



Betti elements

Definition

For an element $n \in S = \langle n_1, \ldots, n_k \rangle$, let ∇_n denote the subgraph of the catenary graph in which only edges (f, f') with $\gcd(f, f') \neq 0$ are drawn.

Betti elements

Definition

For an element $n \in S = \langle n_1, \ldots, n_k \rangle$, let ∇_n denote the subgraph of the catenary graph in which only edges (f, f') with $\gcd(f, f') \neq 0$ are drawn. We say n is a *Betti element* of S if ∇_n is disconnected.

Betti elements

Definition

For an element $n \in S = \langle n_1, \ldots, n_k \rangle$, let ∇_n denote the subgraph of the catenary graph in which only edges (f, f') with $\gcd(f, f') \neq 0$ are drawn. We say n is a *Betti element* of S if ∇_n is disconnected.

Example

$S = \langle 10, 15, 17 \rangle$ has Betti elements 30 and 85.

Betti elements

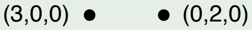
Definition

For an element $n \in S = \langle n_1, \ldots, n_k \rangle$, let ∇_n denote the subgraph of the catenary graph in which only edges (f, f') with $\gcd(f, f') \neq 0$ are drawn. We say n is a *Betti element* of S if ∇_n is disconnected.

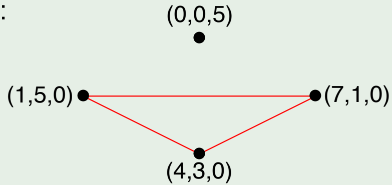
Example

$S = \langle 10, 15, 17 \rangle$ has Betti elements 30 and 85.

∇_{30} :



∇_{85} :



Maximal catenary degree in S

Theorem

$\max\{c(n) : n \in S\} = \max\{c(b) : b \text{ Betti element of } S\}$.

Maximal catenary degree in S

Theorem

$\max\{c(n) : n \in S\} = \max\{c(b) : b \text{ Betti element of } S\}$.

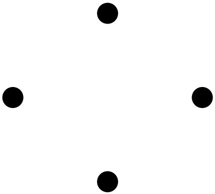
Key concept: Cover morphisms.

Maximal catenary degree in S

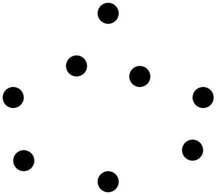
Theorem

$\max\{c(n) : n \in S\} = \max\{c(b) : b \text{ Betti element of } S\}$.

Key concept: Cover morphisms.



$\mathsf{Z}_S(n)$



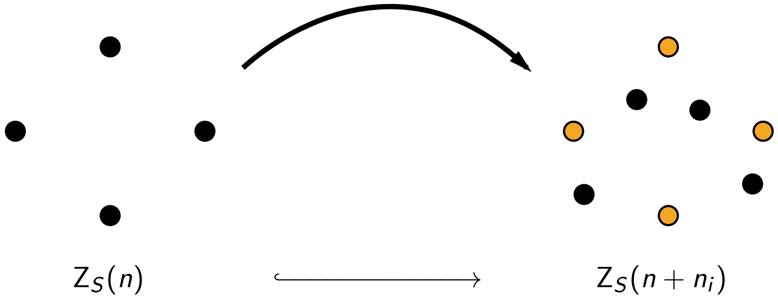
$\mathsf{Z}_S(n + n_i)$

Maximal catenary degree in S

Theorem

$\max\{c(n) : n \in S\} = \max\{c(b) : b \text{ Betti element of } S\}.$

Key concept: Cover morphisms.

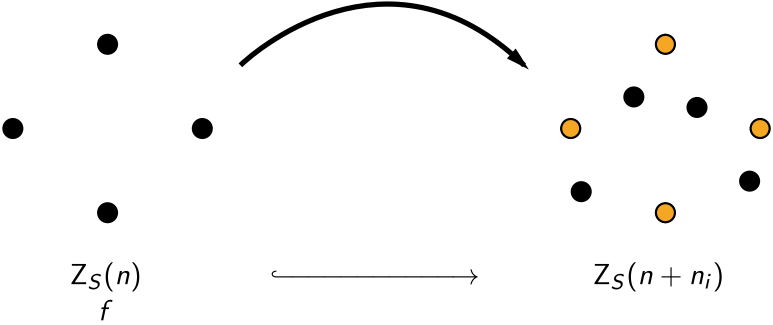


Maximal catenary degree in S

Theorem

$\max\{c(n) : n \in S\} = \max\{c(b) : b \text{ Betti element of } S\}$.

Key concept: Cover morphisms.

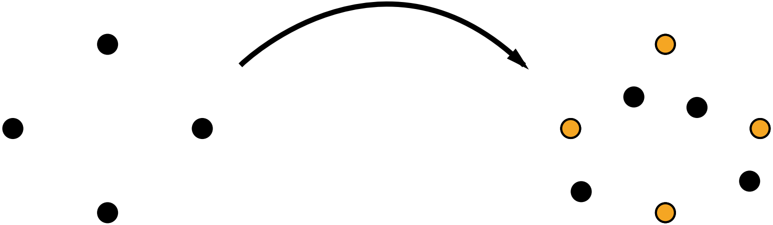


Maximal catenary degree in S

Theorem

$\max\{c(n) : n \in S\} = \max\{c(b) : b \text{ Betti element of } S\}.$

Key concept: Cover morphisms.



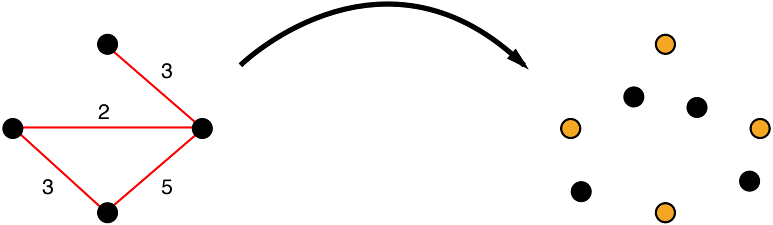
$$\begin{array}{ccc} Z_S(n) & \hookrightarrow & Z_S(n + n_i) \\ f & \longmapsto & f + e_i \end{array}$$

Maximal catenary degree in S

Theorem

$\max\{c(n) : n \in S\} = \max\{c(b) : b \text{ Betti element of } S\}.$

Key concept: Cover morphisms.



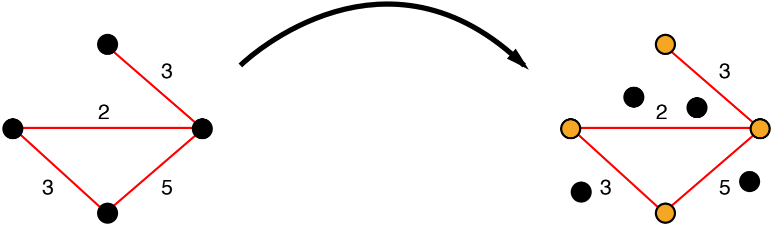
$$Z_S(n) \hookrightarrow Z_S(n + n_i)$$
$$f \longmapsto f + e_i$$

Maximal catenary degree in S

Theorem

$\max\{c(n) : n \in S\} = \max\{c(b) : b \text{ Betti element of } S\}$.

Key concept: Cover morphisms.



$$\begin{aligned} Z_S(n) &\hookrightarrow Z_S(n + n_i) \\ f &\mapsto f + e_i \end{aligned}$$

Maximal catenary degree in S

Theorem

$\max\{c(n) : n \in S\} = \max\{c(b) : b \text{ Betti element of } S\}$.

Maximal catenary degree in S

Theorem

$\max\{c(n) : n \in S\} = \max\{c(b) : b \text{ Betti element of } S\}$.

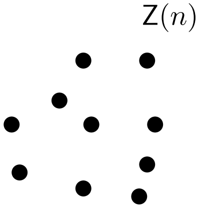
Idea for proof: Certain edges (determined by Betti elements) connect the catenary graph of each $n \in S$.

Maximal catenary degree in S

Theorem

$\max\{c(n) : n \in S\} = \max\{c(b) : b \text{ Betti element of } S\}$.

Idea for proof: Certain edges (determined by Betti elements) connect the catenary graph of each $n \in S$.

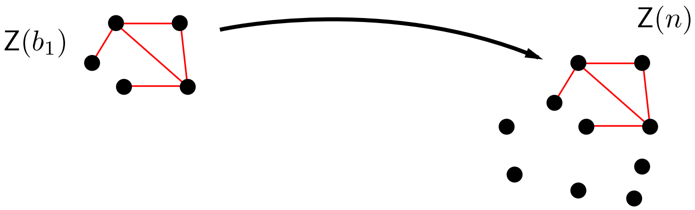


Maximal catenary degree in S

Theorem

$\max\{c(n) : n \in S\} = \max\{c(b) : b \text{ Betti element of } S\}.$

Idea for proof: Certain edges (determined by Betti elements) connect the catenary graph of each $n \in S$.

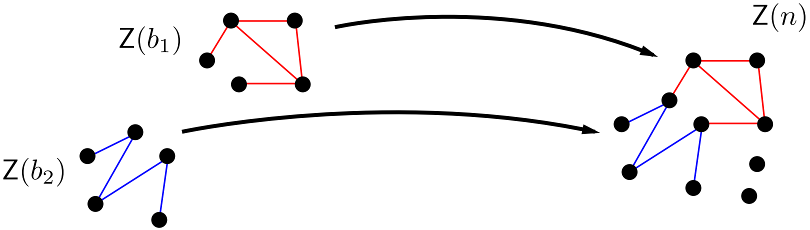


Maximal catenary degree in S

Theorem

$\max\{c(n) : n \in S\} = \max\{c(b) : b$ Betti element of $S\}$.

Idea for proof: Certain edges (determined by Betti elements) connect the catenary graph of each $n \in S$.

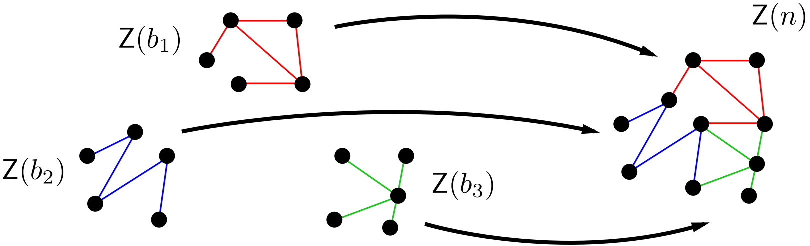


Maximal catenary degree in S

Theorem

$\max\{c(n) : n \in S\} = \max\{c(b) : b \text{ Betti element of } S\}$.

Idea for proof: Certain edges (determined by Betti elements) connect the catenary graph of each $n \in S$.



Minimal (nonzero) catenary degree in S

Conjecture

$\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}.$

Minimal (nonzero) catenary degree in S

~~Conjecture~~ Theorem (O., Ponomarenko, Tate, Webb)

$\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}$.

Minimal (nonzero) catenary degree in S

~~Conjecture~~ Theorem (O., Ponomarenko, Tate, Webb)

$\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}$.

$B = \min\{c(b) : b \text{ Betti element of } S\}$.

Minimal (nonzero) catenary degree in S

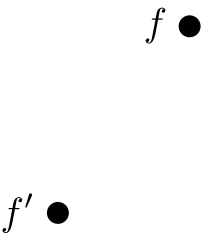
~~Conjecture~~ Theorem (O., Ponomarenko, Tate, Webb)

$\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}$.

$B = \min\{c(b) : b \text{ Betti element of } S\}$.

Lemma

If $f, f' \in \mathsf{Z}_S(n)$



Minimal (nonzero) catenary degree in S

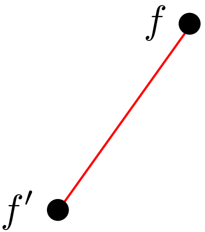
~~Conjecture~~ Theorem (O., Ponomarenko, Tate, Webb)

$\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}$.

$B = \min\{c(b) : b \text{ Betti element of } S\}$.

Lemma

If $f, f' \in \mathsf{Z}_S(n)$ and $d(f, f') < B$,



Minimal (nonzero) catenary degree in S

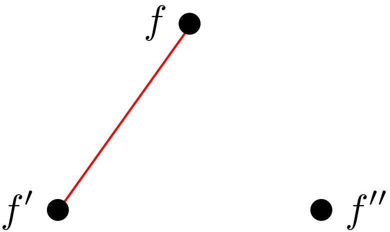
~~Conjecture~~ Theorem (O., Ponomarenko, Tate, Webb)

$\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}$.

$B = \min\{c(b) : b \text{ Betti element of } S\}$.

Lemma

If $f, f' \in Z_S(n)$ and $d(f, f') < B$, then there exists $f'' \in Z_S(n)$



Minimal (nonzero) catenary degree in S

~~Conjecture~~ Theorem (O., Ponomarenko, Tate, Webb)

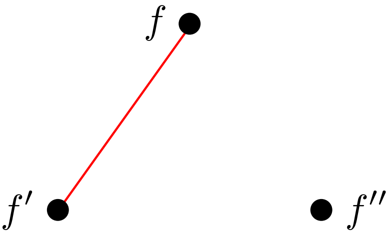
$\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}$.

$B = \min\{c(b) : b \text{ Betti element of } S\}$.

Lemma

If $f, f' \in Z_S(n)$ and $d(f, f') < B$, then there exists $f'' \in Z_S(n)$ with

$$\max\{|f|, |f'|\} < |f''|.$$



Minimal (nonzero) catenary degree in S

~~Conjecture~~ Theorem (O., Ponomarenko, Tate, Webb)

$\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}$.

Minimal (nonzero) catenary degree in S

~~Conjecture~~ Theorem (O., Ponomarenko, Tate, Webb)

$\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}$.

Proof of theorem:

Minimal (nonzero) catenary degree in S

~~Conjecture~~ Theorem (O., Ponomarenko, Tate, Webb)

$\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}$.

Proof of theorem:

- Fix $n \in S$

Minimal (nonzero) catenary degree in S

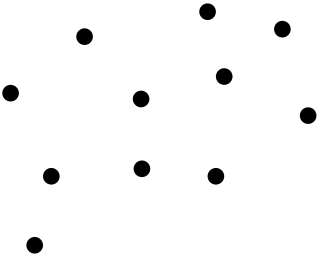
~~Conjecture~~ Theorem (O., Ponomarenko, Tate, Webb)

$\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}$.

Proof of theorem:

- Fix $n \in S$

Catenary graph of n:



Minimal (nonzero) catenary degree in S

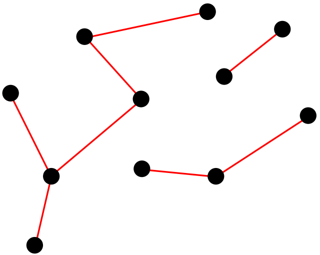
~~Conjecture~~ Theorem (O., Ponomarenko, Tate, Webb)

$\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}$.

Proof of theorem:

- Fix $n \in S$
- Draw edges with weight $< B$

Catenary graph of n:



Minimal (nonzero) catenary degree in S

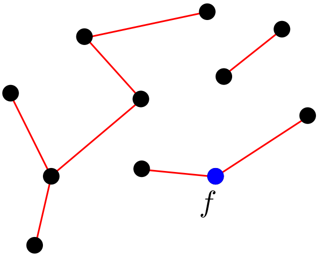
~~Conjecture~~ Theorem (O., Ponomarenko, Tate, Webb)

$\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}$.

Proof of theorem:

- Fix $n \in S$
- Draw edges with weight $< B$
- $f \in \mathsf{Z}_S(n)$ with $|f|$ maximal

Catenary graph of n:



Minimal (nonzero) catenary degree in S

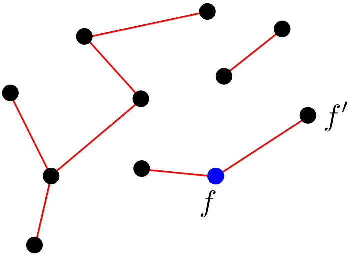
~~Conjecture~~ Theorem (O., Ponomarenko, Tate, Webb)

$\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}$.

Proof of theorem:

- Fix $n \in S$
- Draw edges with weight $< B$
- $f \in \mathsf{Z}_S(n)$ with $|f|$ maximal
- $f' \in \mathsf{Z}_S(n)$ with $d(f, f') < B$

Catenary graph of n:



Minimal (nonzero) catenary degree in S

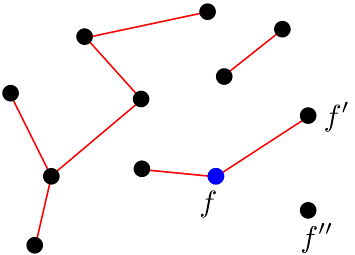
~~Conjecture~~ Theorem (O., Ponomarenko, Tate, Webb)

$\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}$.

Proof of theorem:

- Fix $n \in S$
- Draw edges with weight $< B$
- $f \in \mathsf{Z}_S(n)$ with $|f|$ maximal
- $f' \in \mathsf{Z}_S(n)$ with $d(f, f') < B$
- Lemma $\Rightarrow |f''| > |f|$

Catenary graph of n:



Minimal (nonzero) catenary degree in S

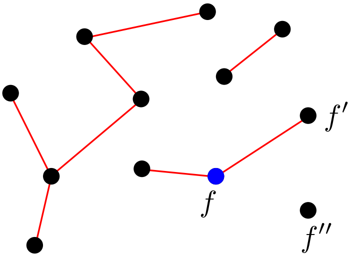
~~Conjecture~~ Theorem (O., Ponomarenko, Tate, Webb)

$\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \text{ Betti element of } S\}$.

Proof of theorem:

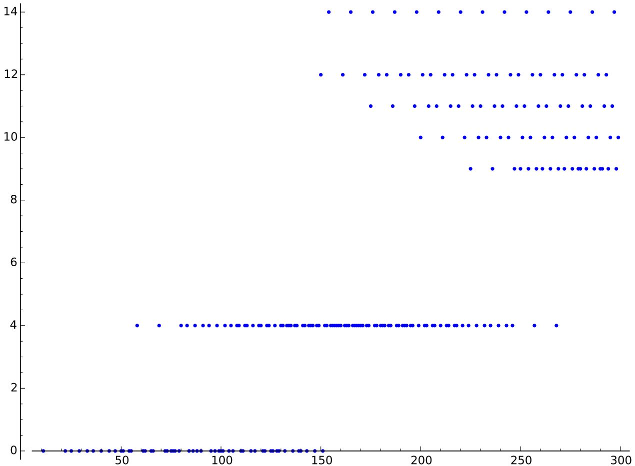
- Fix $n \in S$
- Draw edges with weight $< B$
- $f \in Z_S(n)$ with $|f|$ maximal
- $f' \in Z_S(n)$ with $d(f, f') < B$
- Lemma $\Rightarrow |f''| > |f|$
- maximality of $|f| \Rightarrow$
  f'' has no edges!

Catenary graph of n:

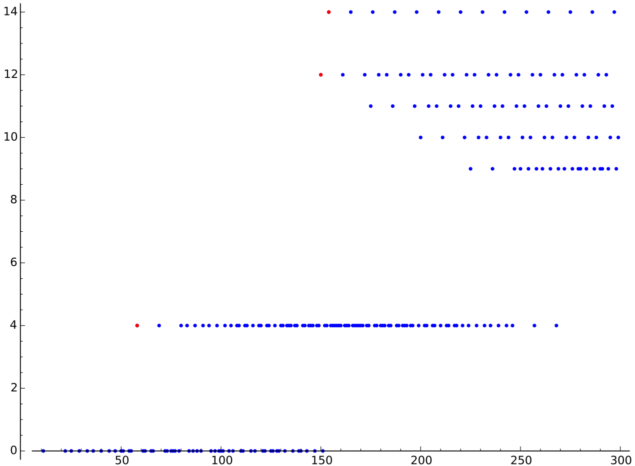


Future directions: catenary sets

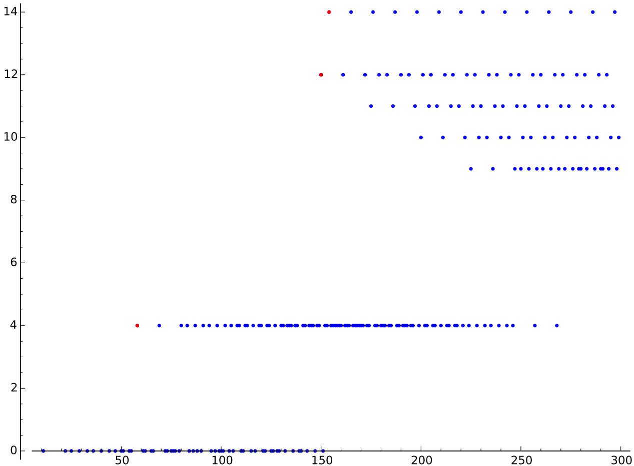
Future directions: catenary sets



Future directions: catenary sets



Future directions: catenary sets



Problem

Find a (canonical) finite set on which every catenary degree is achieved.

References

Alfred Geroldinger, Franz Halter-Koch (2006)
Nonunique factorization: Algebraic, Combinatorial, and Analytic Theory.
Chapman & Hall/CRC, Boca Raton, FL, 2006.

Scott Champan, Pedro García-Sánchez, David Llena, Vadim Ponomarenko, José Rosales (2006)
The catenary and tame degree in finitely generated cancellative commutative monoids.
Manus. Math, 120 (2006) 253 – 264.

Christopher O'Neill, Vadim Ponomarenko, Reuben Tate, Gautam Webb (2014)
On the set of catenary degrees in numerical monoids.
In preparation.

Manuel Delgado, Pedro García-Sánchez, Jose Morais
GAP Numerical Semigroups Package
`http://www.gap-system.org/Packages/numericalsgps.html`.

References

Alfred Geroldinger, Franz Halter-Koch (2006)
Nonunique factorization: Algebraic, Combinatorial, and Analytic Theory.
Chapman & Hall/CRC, Boca Raton, FL, 2006.

Scott Champan, Pedro García-Sánchez, David Llena, Vadim Ponomarenko, José Rosales (2006)
The catenary and tame degree in finitely generated cancellative commutative monoids.
Manus. Math, 120 (2006) 253 – 264.

Christopher O'Neill, Vadim Ponomarenko, Reuben Tate, Gautam Webb (2014)
On the set of catenary degrees in numerical monoids.
In preparation.

Manuel Delgado, Pedro García-Sánchez, Jose Morais
GAP Numerical Semigroups Package
`http://www.gap-system.org/Packages/numericalsgps.html`.

Thanks!