Computing the delta set and ω-primality in numerical monoids

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Joint with Thomas Barron and Roberto Pelayo

June 13, 2015

The delta set

Definition (Numerical monoid)

A *numerical monoid* S is an additive submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

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Goal

Compute $\Delta(S) = \bigcup_{n \in S} \Delta(n)$.

Computing the delta set of a numerical monoid

Theorem (Chapman–Hoyer–Kaplan, 2000)

$S = \langle n_1, \ldots, n_k \rangle$. *For* $n \geq 2kn_2n_k^2$, $\Delta(n) = \Delta(n + n_1n_k)$.

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Theorem (García-García–Moreno-Frías–Vigneron-Tenorio, 2014)

$S = \langle n_1, \ldots, n_k \rangle$. For $n \geq N_S$, $\Delta(n) = \Delta(n + \mathrm{lcm}(n_1, n_k))$.

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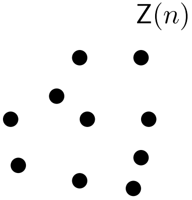

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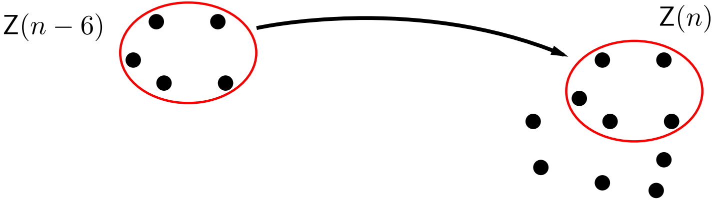

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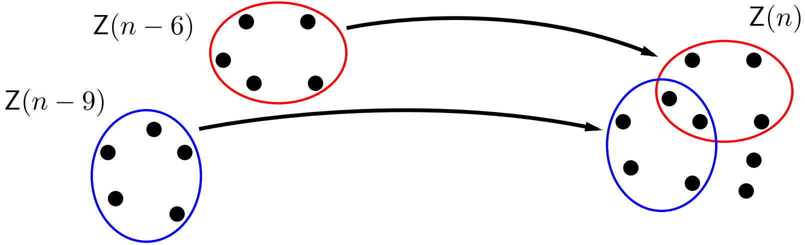

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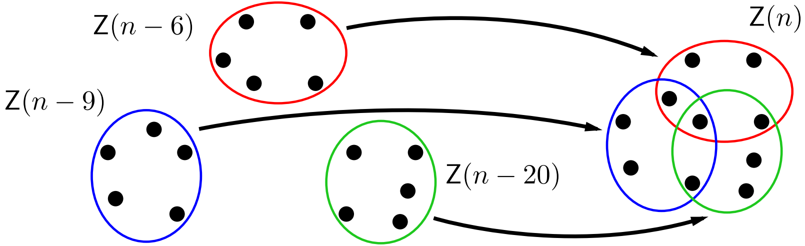

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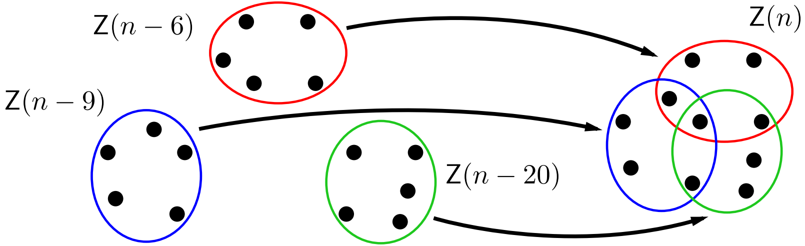

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18	$2\mathbf{e}_1 \overset{6}{\rightsquigarrow} 3\mathbf{e}_1$	$\{3\mathbf{e}_1, 2\mathbf{e}_2\}$	$\{2, 3\}$
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20	$\mathbf{0} \overset{20}{\rightsquigarrow} \mathbf{e}_3$	$\{\mathbf{e}_3\}$	$\{1\}$

A solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \le k$,

$$\begin{array}{rcl} \phi_i : \mathsf{Z}(n - n_i) & \longrightarrow & \mathsf{Z}(n) \\ \mathbf{a} & \longmapsto & \mathbf{a} + \mathbf{e}_i \end{array}$$

$$\mathsf{Z}(n) = \bigcup_{i \le k} \phi_i(\mathsf{Z}(n - n_i))$$

$n \in S = \langle 6, 9, 20 \rangle$		$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0		$\{\mathbf{0}\}$	$\{0\}$
6	$\mathbf{0} \overset{6}{\rightsquigarrow} \mathbf{e}_1$	$\{\mathbf{e}_1\}$	$\{1\}$
9	$\mathbf{0} \overset{9}{\rightsquigarrow} \mathbf{e}_2$	$\{\mathbf{e}_2\}$	$\{1\}$
12	$\mathbf{e}_1 \overset{6}{\rightsquigarrow} 2\mathbf{e}_1$	$\{2\mathbf{e}_1\}$	$\{2\}$
15	$\mathbf{e}_2 \overset{6}{\rightsquigarrow} (1,1,0)$	$\{(1,1,0)\}$	$\{2\}$
	$\mathbf{e}_1 \overset{9}{\rightsquigarrow} (1,1,0)$		
18	$2\mathbf{e}_1 \overset{6}{\rightsquigarrow} 3\mathbf{e}_1$	$\{3\mathbf{e}_1, 2\mathbf{e}_2\}$	$\{2, 3\}$
	$\mathbf{e}_2 \overset{9}{\rightsquigarrow} 2\mathbf{e}_2$		
20	$\mathbf{0} \overset{20}{\rightsquigarrow} \mathbf{e}_3$	$\{\mathbf{e}_3\}$	$\{1\}$
$\vdots$		$\vdots$	$\vdots$

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$$\mathsf{Z}(n) = \textstyle\bigcup_{i \le k} \phi_i(\mathsf{Z}(n - n_i))$$

$n \in S = \langle 6, 9, 20 \rangle$		$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0		$\{\mathbf{0}\}$	$\{0\}$
6	$\mathbf{0} \overset{6}{\rightsquigarrow} \mathbf{e}_1$	$\{\mathbf{e}_1\}$	$\{1\}$
9	$\mathbf{0} \overset{9}{\rightsquigarrow} \mathbf{e}_2$	$\{\mathbf{e}_2\}$	$\{1\}$
12	$\mathbf{e}_1 \overset{6}{\rightsquigarrow} 2\mathbf{e}_1$	$\{2\mathbf{e}_1\}$	$\{2\}$
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18	$2\mathbf{e}_1 \overset{6}{\rightsquigarrow} 3\mathbf{e}_1$	$\{3\mathbf{e}_1, 2\mathbf{e}_2\}$	$\{2, 3\}$
	$\mathbf{e}_2 \overset{9}{\rightsquigarrow} 2\mathbf{e}_2$		
20	$\mathbf{0} \overset{20}{\rightsquigarrow} \mathbf{e}_3$	$\{\mathbf{e}_3\}$	$\{1\}$
$\vdots$		$\vdots$	$\vdots$

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$n \in S = \langle 6, 9, 20 \rangle$		$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0		$\{\mathbf{0}\}$	$\{0\}$
6	$\mathbf{0} \overset{6}{\rightsquigarrow} \mathbf{e}_1$	$\{\mathbf{e}_1\}$	$\{1\}$
9	$\mathbf{0} \overset{9}{\rightsquigarrow} \mathbf{e}_2$	$\{\mathbf{e}_2\}$	$\{1\}$
12	$\mathbf{e}_1 \overset{6}{\rightsquigarrow} 2\mathbf{e}_1$	$\{2\mathbf{e}_1\}$	$\{2\}$
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$n \in S = \langle 6, 9, 20 \rangle$		$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0		$\{\mathbf{0}\}$	$\{0\}$
6	$\mathbf{0} \overset{6}{\rightsquigarrow} \mathbf{e}_1$	$\{\mathbf{e}_1\}$	$\{1\}$
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$\vdots$		$\vdots$	$\vdots$

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$$\mathbf{a} \longmapsto \mathbf{a} + \mathbf{e}_i \qquad \ell \longmapsto \ell + 1$$

$$\mathsf{Z}(n) = \bigcup_{i \le k} \phi_i(\mathsf{Z}(n - n_i)) \qquad \mathsf{L}(n) = \bigcup_{i \le k} \psi_i(\mathsf{L}(n - n_i))$$

$n \in S = \langle 6, 9, 20 \rangle$	$\mathsf{L}(n)$
0	$\{0\}$
6	
9	
12	
15	
18	
20	
$\vdots$	

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$n \in S = \langle 6, 9, 20 \rangle$	$\mathsf{L}(n)$	
0	$\{0\}$	
6	$\{1\}$	$0 \overset{6}{\rightsquigarrow} 1$
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0	$\{0\}$	
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20		
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Computing the delta set dynamically

Theorem (García-García–Moreno-Frías–Vigneron-Tenorio, 2014)

$S = \langle n_1, \ldots, n_k \rangle$. *For* $n \geq N_S$, $\Delta(n) = \Delta(n + \mathrm{lcm}(n_1, n_k))$.

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compute:

$$\mathsf{Z}(n) = \{\mathbf{a} \in \mathbb{N}^k : n = a_1 n_1 + \cdots + a_k n_k\}$$
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$$|\mathsf{L}(n)| \approx n$$

Runtime comparison

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S	N_S	$\Delta(S)$	Existing	Dynamic
$\langle 7, 15, 17, 18, 20 \rangle$	1935	$\{1, 2, 3\}$	1m 28s	146ms
$\langle 11, 53, 73, 87 \rangle$	14381	$\{2, 4, 6, 8, 10, 22\}$	0m 49s	2.5s
$\langle 31, 73, 77, 87, 91 \rangle$	31364	$\{2, 4, 6\}$	400m 12s	4.2s
$\langle 100, 121, 142, 163, 284 \rangle$	24850	$\{21\}$	———	0m 3.6s
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GAP Numerical Semigroups Package, available at

`http://www.gap-system.org/Packages/numericalsgps.html`.

ω-primality

As usual, $n \in S = \langle n_1, \ldots, n_k \rangle$.

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$\omega_S(n)$ is the minimal m such that whenever $(\sum_{i=1}^{r} x_i) - n \in S$ for $r > m$, there exists $T \subset \{1, \ldots, r\}$ with $|T| \leq m$ and $(\sum_{i \in T} x_i) - n \in S$.

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A *bullet* for $n \in S$ is a tuple $\mathbf{b} = (b_1, \ldots, b_k) \in \mathbb{N}^k$ such that

(i) $b_1 n_1 + \cdots + b_k n_k - n \in S$, and

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The set of bullets of n is denoted $\mathrm{bul}(n)$.

ω-primality

As usual, $n \in S = \langle n_1, \ldots, n_k \rangle$.

Definition (ω-primality)

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Proposition

$\omega_S(n) = \max\{|\mathbf{b}| : \mathbf{b} \in \text{bul}(n)\}$.

Using bullets to compute ω-primality

Algorithm: Compute $\text{bul}(n)$, then compute $\omega(n) = \max\{|\mathbf{b}| : \mathbf{b} \in \text{bul}(n)\}$.

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Algorithm: Compute $\mathsf{bul}(n)$, then compute $\omega(n) = \max\{|\mathbf{b}| : \mathbf{b} \in \mathsf{bul}(n)\}$.

Example

$S = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$.

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$$\mathrm{bul}(60) = \{(4,4,0), (7,2,0), (10,0,0), (1,6,0), (0,8,0), (0,0,3)\}$$

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$n \in S$	$\omega(n)$	mbul
6	3	$3\mathbf{e}_3$
9	3	$3\mathbf{e}_3$
12	3	$3\mathbf{e}_3$

$n \in S$	$\omega(n)$	mbul
15	4	$4\mathbf{e}_1$
18	3	$3\mathbf{e}_1$
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$n \in S$	$\omega(n)$	mbul
21	5	$5\mathbf{e}_1$
24	4	$4\mathbf{e}_1$
26	11	$11\mathbf{e}_1$

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Moral of this talk: bullets behave like factorizations!

Toward a dynamic algorithm... the inductive step

Recall: for $n \in S = \langle n_1, \dots n_k \rangle$, $\mathsf{Z}(n) = \{\mathbf{a} \in \mathbb{N}^k : \sum_{i=1}^k a_i n_i = n\}$.

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$$\begin{aligned} \phi_i : \mathsf{Z}(n - n_i) &\longrightarrow \mathsf{Z}(n) \\ \mathbf{a} &\longmapsto \mathbf{a} + \mathbf{e}_i. \end{aligned}$$

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Fix $n \in S$ and $i \leq k$. The *i-th cover morphism for n* is the map

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Moreover, $\mathsf{bul}(n) = \bigcup_{i \le k} \psi_i(\mathsf{bul}(n - n_i))$.**

Toward a dynamic algorithm. . . the base case

Definition (ω-primality in numerical monoids)

Fix a numerical monoid S and $n \in S$.

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Remark

All properties of ω extend from S to $\mathbb{Z}$.

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Remark

All properties of ω extend from S to $\mathbb{Z}$.

Proposition

For $n \in \mathbb{Z}$, the following are equivalent:

(i) $\omega(n) = 0$,
(ii) $\mathsf{bul}(n) = \{\mathbf{0}\}$,
(iii) $-n \in S$.

A dynamic algorithm!

Example

$S = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$.

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$n \in \mathbb{Z}$	$\omega(n)$	$\mathsf{bul}(n)$
≤ -44	0	$\{\mathbf{0}\}$

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$S = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$.

$n \in \mathbb{Z}$	$\omega(n)$	$\mathsf{bul}(n)$
≤ -44	0	$\{\mathbf{0}\}$
-43	1	$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$
-42	0	$\{\mathbf{0}\}$
$\vdots$	$\vdots$	$\vdots$
-38	0	$\{\mathbf{0}\}$
-37	2	$\{2\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$
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-33	0	$\{\mathbf{0}\}$
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148	28	$\{28\mathbf{e}_1, \ldots\}$

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$\vdots$	$\vdots$	$\vdots$
148	28	$\{28\mathbf{e}_1, \ldots\}$
149	33	$\{33\mathbf{e}_1, \ldots\}$

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148	28	$\{28\mathbf{e}_1, \ldots\}$
149	33	$\{33\mathbf{e}_1, \ldots\}$
150	25	$\{25\mathbf{e}_1, \ldots\}$

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$n \in \mathbb{Z}$	$\omega(n)$	$\mathrm{bul}(n)$
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9	3	$\{3\mathbf{e}_1, 3\mathbf{e}_3, \ldots\}$
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150	25	$\{25\mathbf{e}_1, \ldots\}$

Runtime comparison

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S	$n \in S$	$\omega_S(n)$	Existing	Dynamic
$\langle 6, 9, 20 \rangle$	1000	170	1m 1.3s	6ms
$\langle 11, 13, 15 \rangle$	1000	97	0m 10.7s	5ms
$\langle 11, 13, 15 \rangle$	3000	279	14m 34.7s	15ms
$\langle 11, 13, 15 \rangle$	10000	915	———	42ms
$\langle 15, 27, 32, 35 \rangle$	1000	69	3m 54.7s	9ms
$\langle 100, 121, 142, 163, 284 \rangle$	25715	308	———	0m 27s
$\langle 1001, 1211, 1421, 1631, 2841 \rangle$	357362	405	———	57m 27s

GAP Numerical Semigroups Package, available at

`http://www.gap-system.org/Packages/numericalsgps.html`.

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