Computing the catenary degree, delta set, and omega-primality in numerical monoids

Christopher O'Neill

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July 13, 2015

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First half: Catenary degree
Joint with Vadim Ponomarenko, Reuben Tate*, and Gautam Webb*

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Second half: delta sets and omega-primality
Joint with Thomas Barron* and Roberto Pelayo

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Factorization invariants: towards the catenary degree

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Fix a numerical monoid $S = \langle n_1, \ldots, n_k \rangle$. For $n \in S$,

$$\mathsf{Z}_S(n) = \left\{ \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k : n = a_1 n_1 + \cdots + a_k n_k \right\}$$

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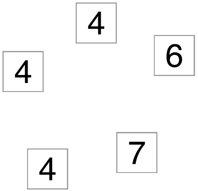

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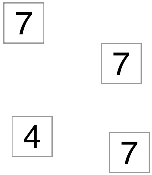

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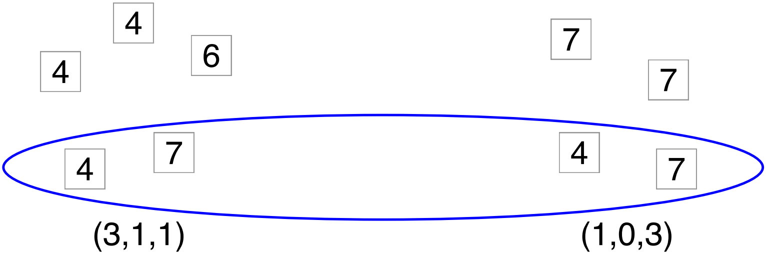

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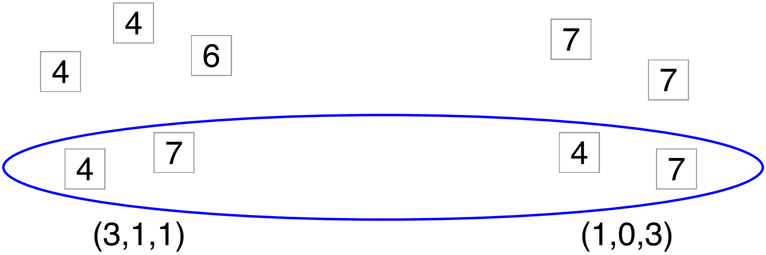

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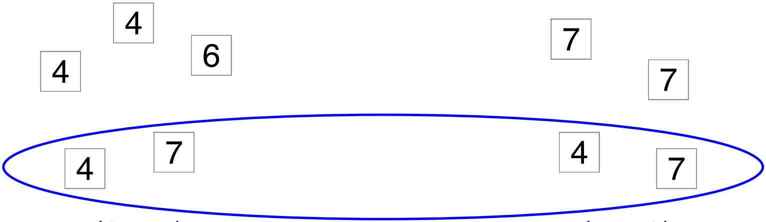

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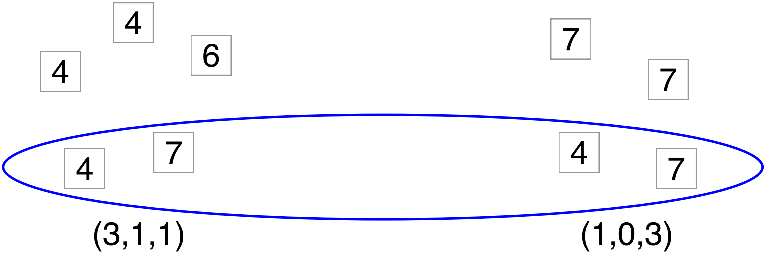

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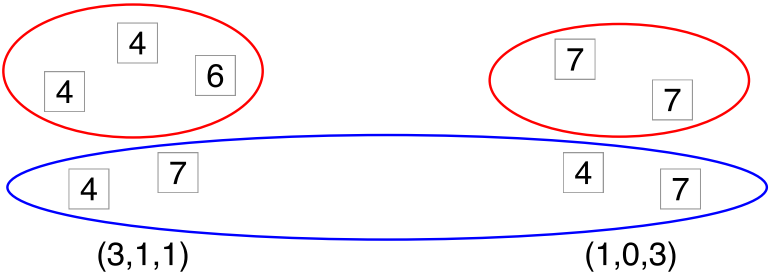

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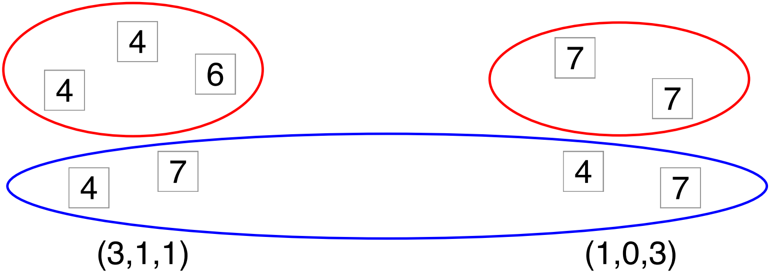

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If $|\mathsf{Z}_S(n)| = 1$, define $c(n) = 0$.

A Big Example

$S = \langle 11, 36, 39 \rangle, n = 450$

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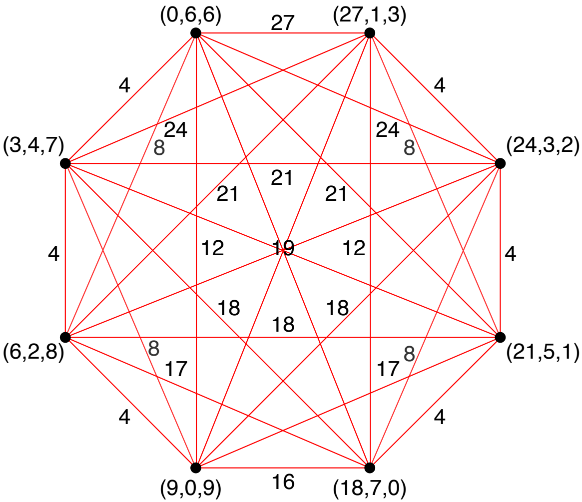

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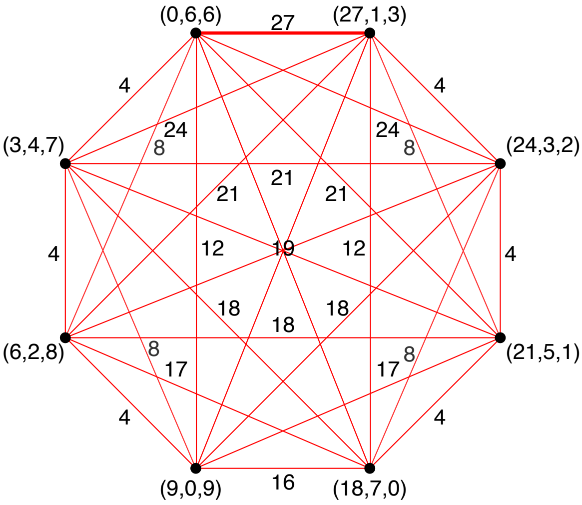

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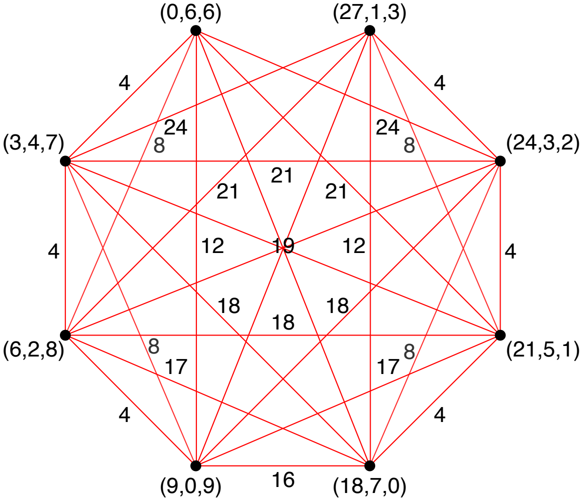

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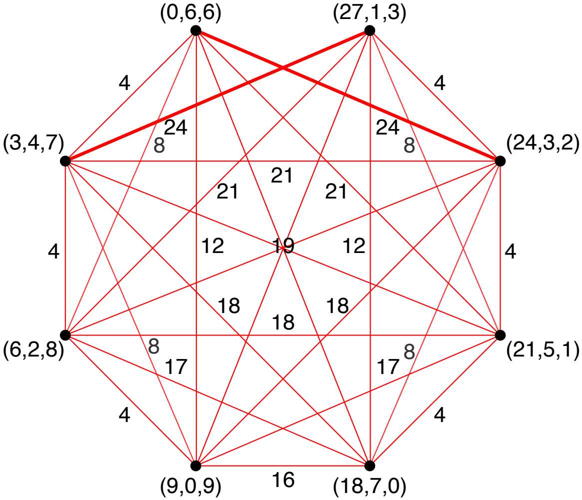

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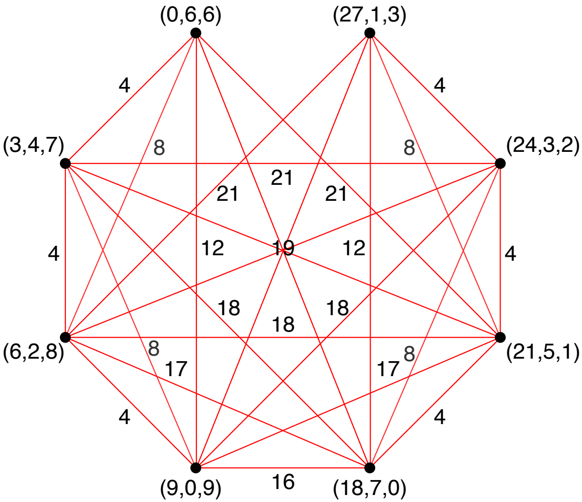

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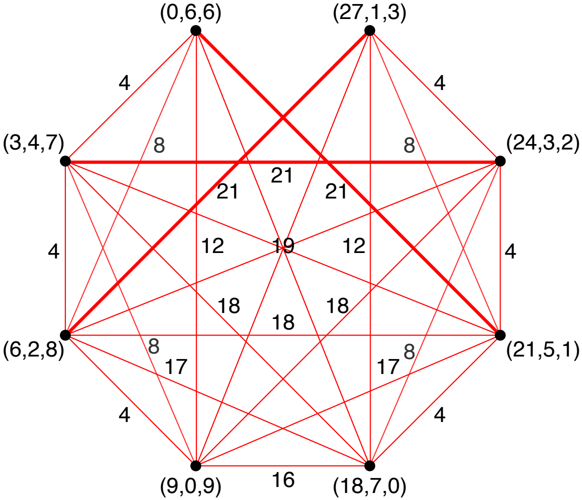

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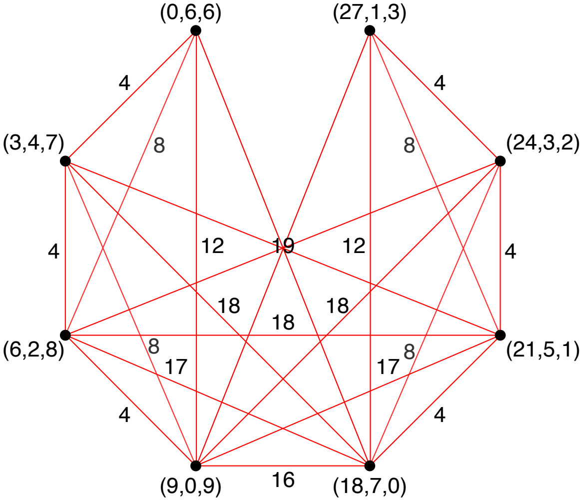

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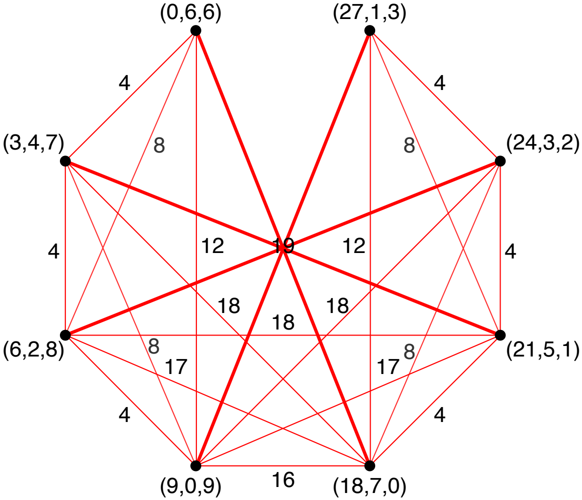

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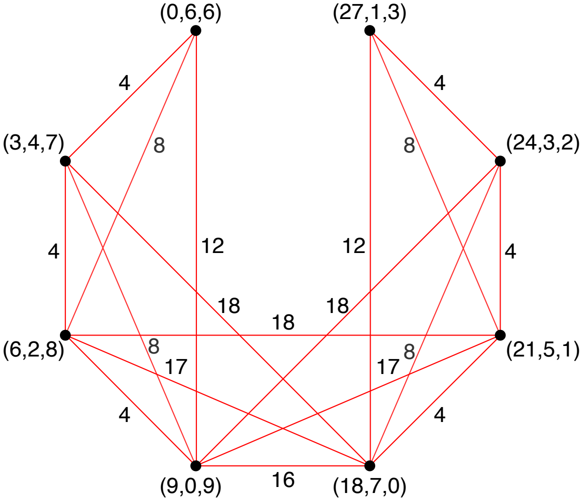

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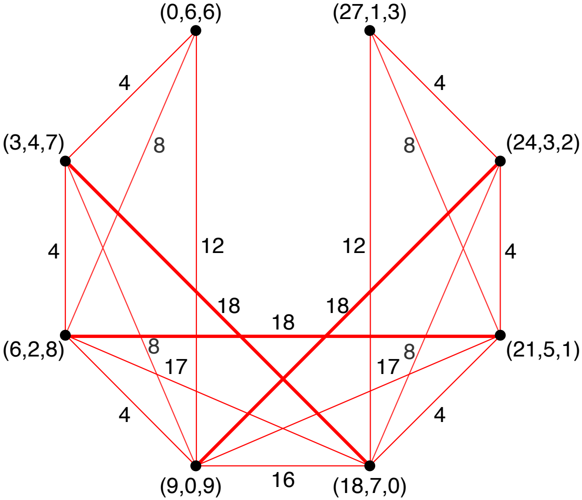

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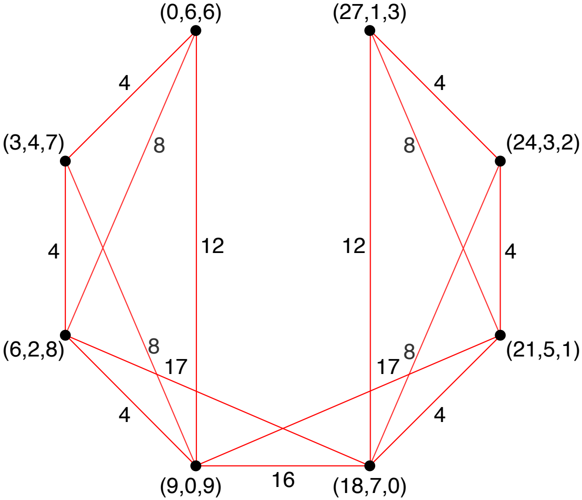

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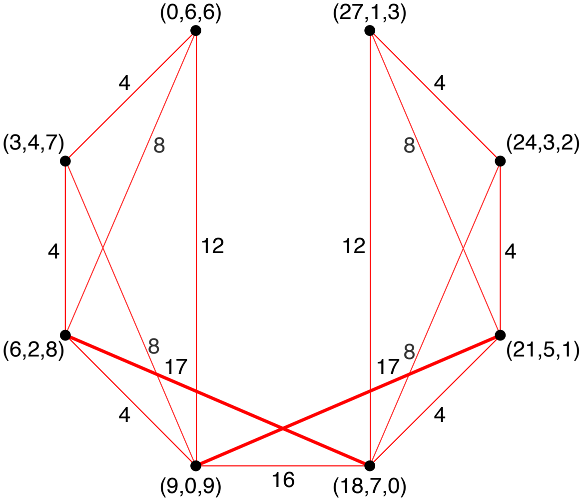

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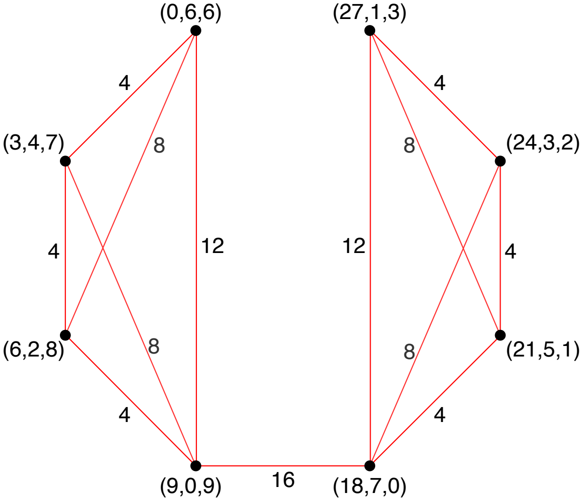

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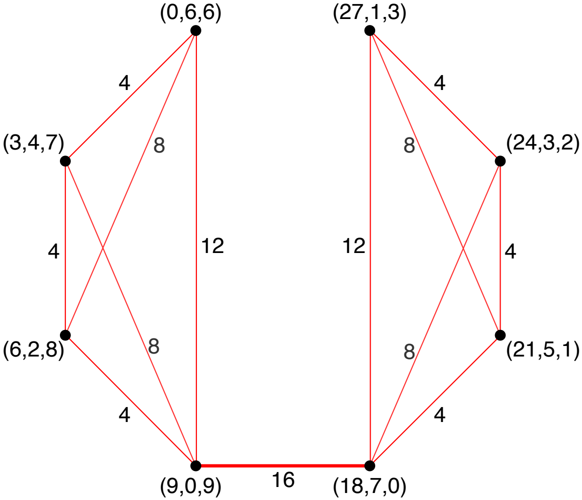

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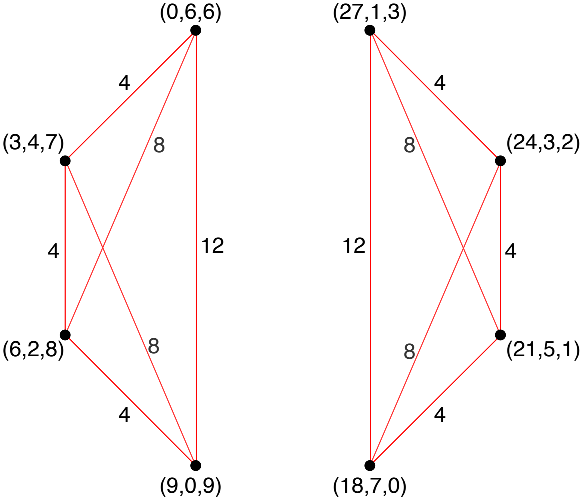

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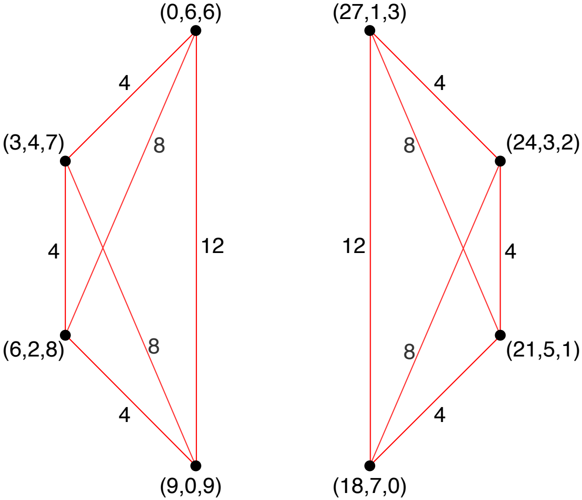
$S = \langle 11, 36, 39 \rangle$, $n = 450$, $c(n) = 16$



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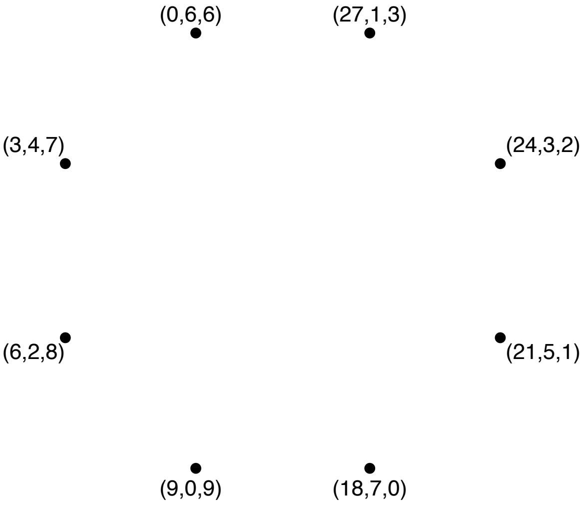

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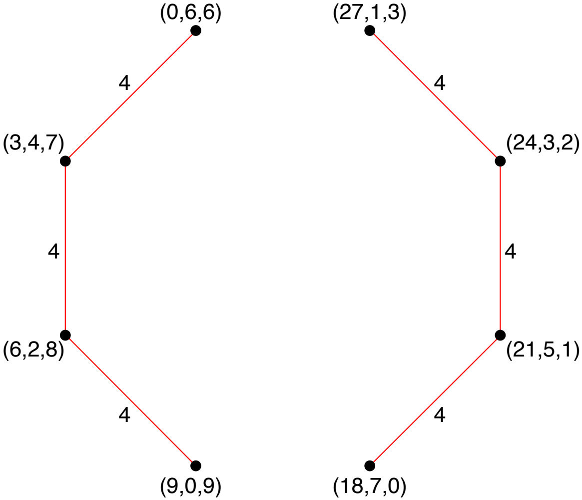

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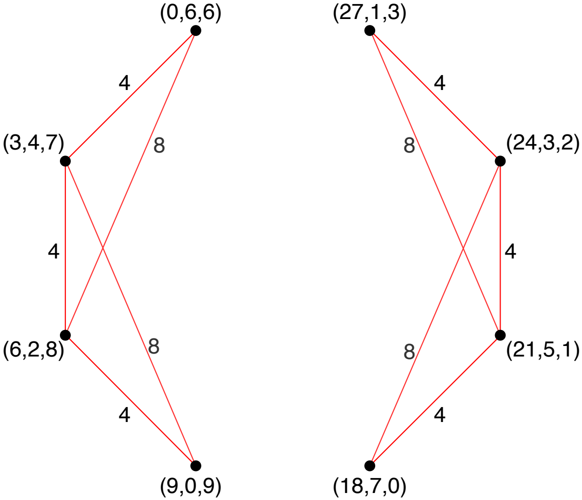

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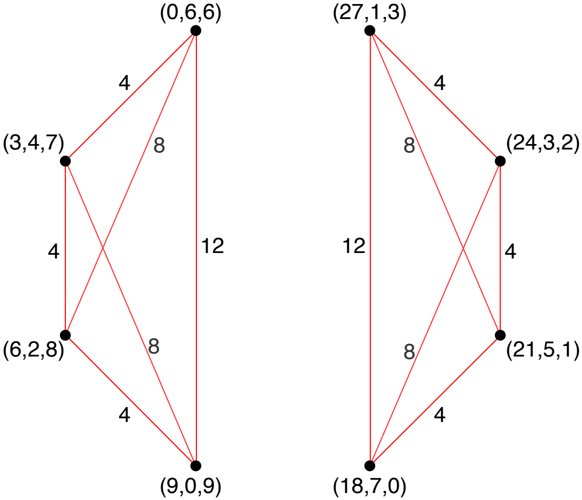

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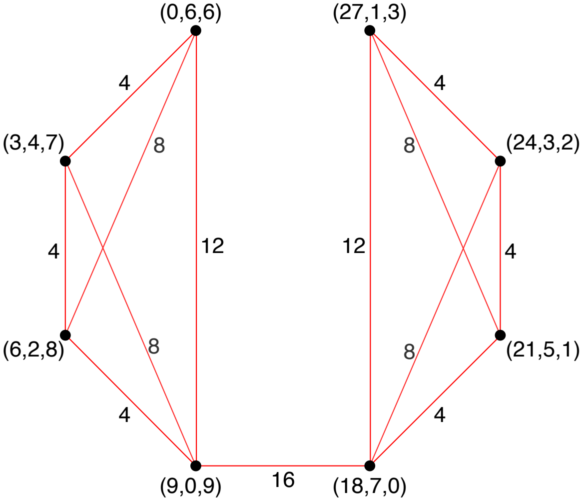

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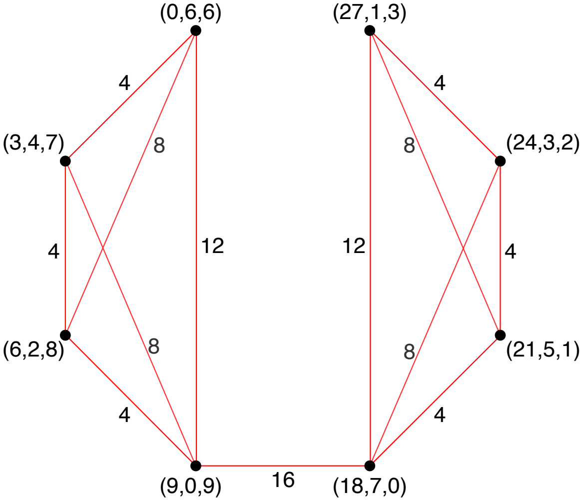

Betti elements

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For an element $n \in S = \langle n_1, \ldots, n_k \rangle$, let ∇_n denote the subgraph of the catenary graph in which only edges $(\mathbf{a}, \mathbf{a}')$ with $\gcd(\mathbf{a}, \mathbf{a}') \neq 0$ are drawn.

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$S = \langle 10, 15, 17 \rangle$ has Betti elements 30 and 85.

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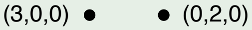

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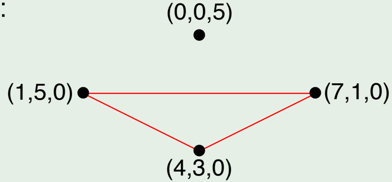

Maximal catenary degree in S

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Key concept: Cover morphisms.

Maximal catenary degree in S

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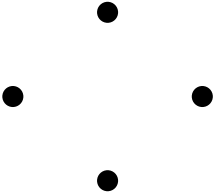

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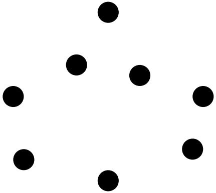

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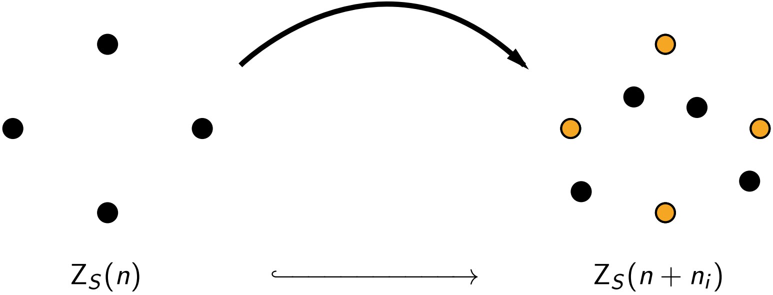

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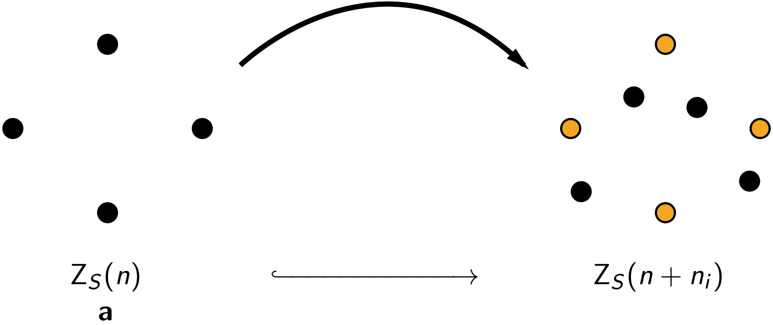

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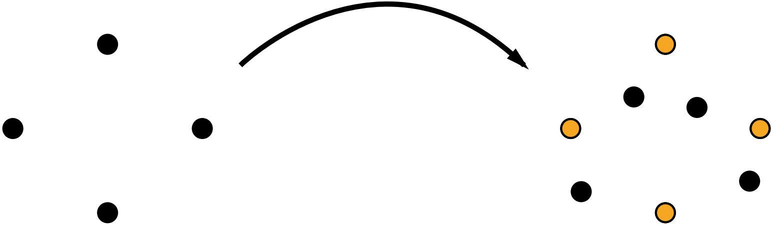

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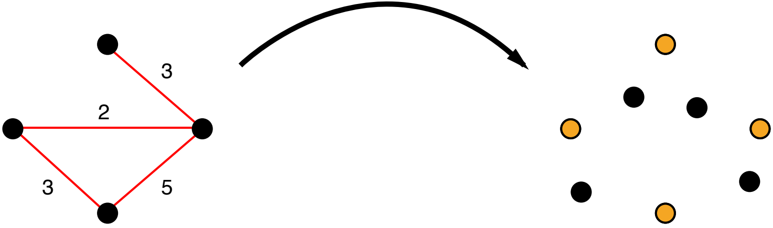

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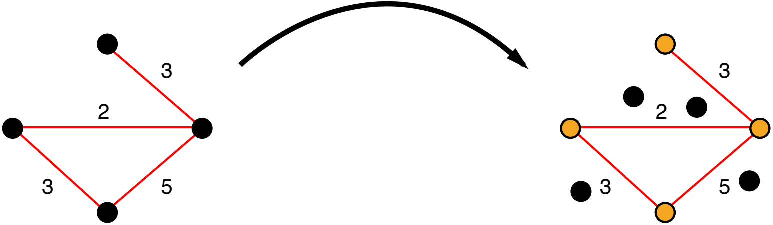

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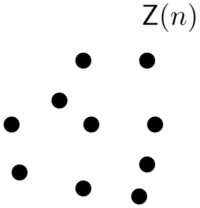

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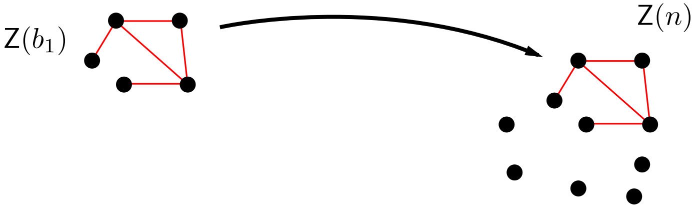

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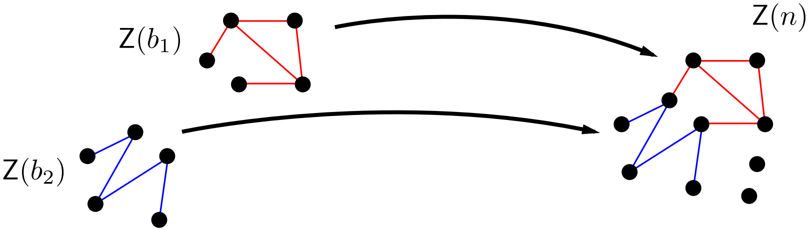

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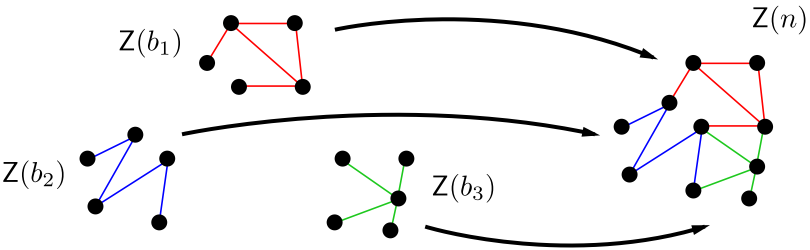

Minimal (nonzero) catenary degree in S

Conjecture

$\min\{c(n) > 0 : n \in S\} = \min\{c(b) : b \in \mathrm{Betti}(S)\}$.

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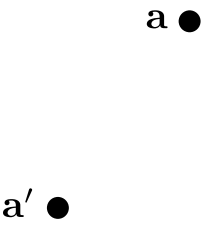

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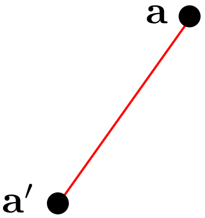

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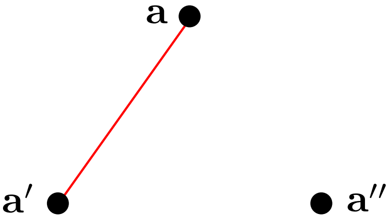

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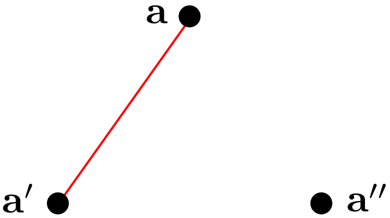

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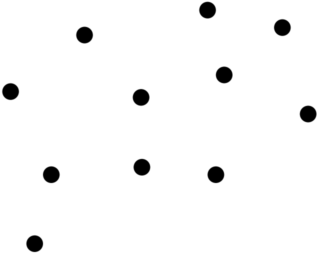

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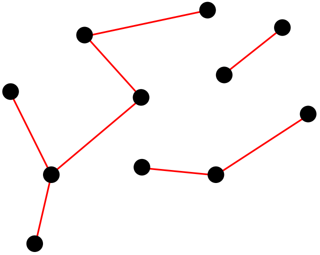

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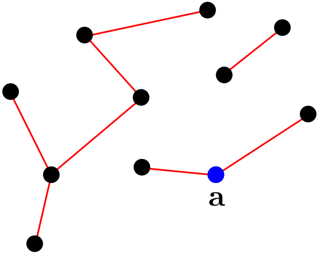

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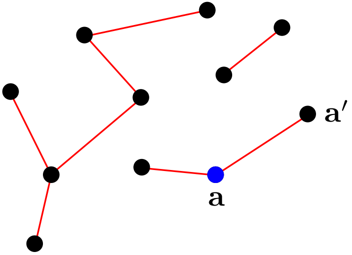

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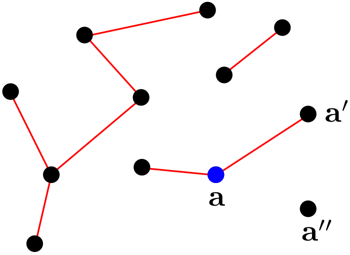

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  $\mathbf{a}''$ has no red edges!

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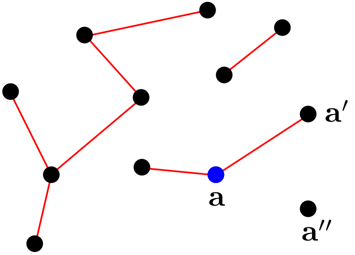

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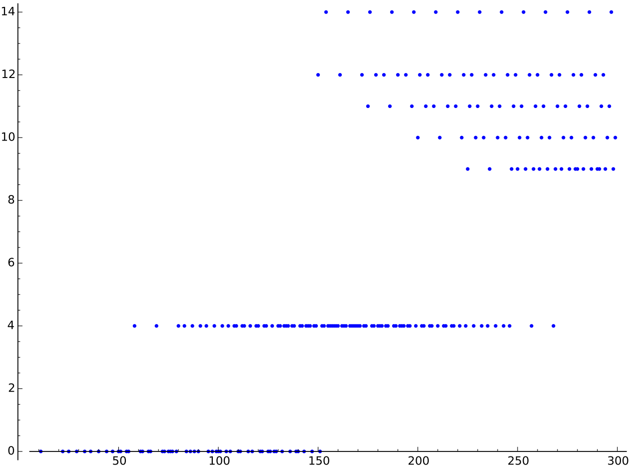

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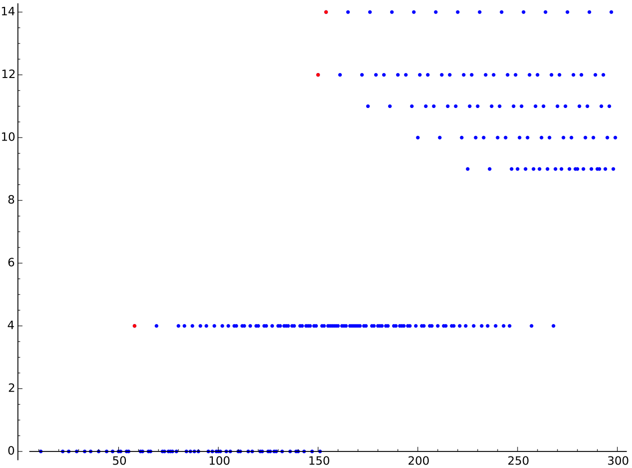

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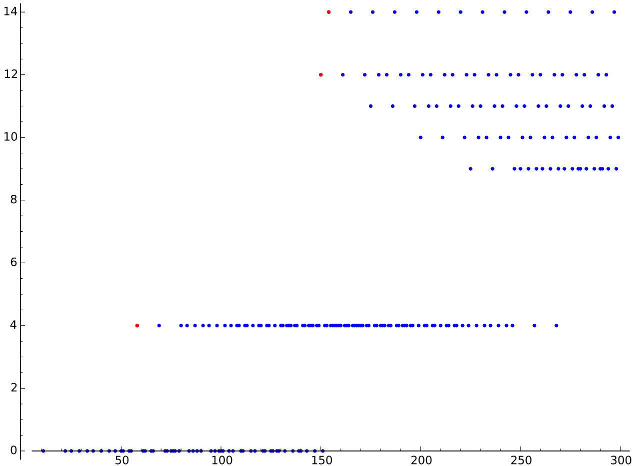

Problem

Find a (canonical) finite set on which every catenary degree is achieved.

The delta set

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$.

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For $\mathsf{L}(n) = \{\ell_1 < \ldots < \ell_r\}$, define $\Delta(n) = \{\ell_i - \ell_{i-1}\}$

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Goal

Compute $\Delta(S) = \bigcup_{n \in S} \Delta(n)$.

Computing the delta set of a numerical monoid

Theorem (Chapman–Hoyer–Kaplan, 2000)

$S = \langle n_1, \ldots, n_k \rangle$. For $n \geq 2kn_2n_k^2$, $\Delta(n) = \Delta(n + n_1 n_k)$.

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Theorem (García-García–Moreno-Frías–Vigneron-Tenorio, 2014)

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$$\mathsf{Z}(n) = \{\mathbf{a} \in \mathbb{N}^k : n = a_1 n_1 + \cdots + a_k n_k\}$$
$$\mathsf{Z}(n) \rightsquigarrow \mathsf{L}(n) = \{a_1 + \cdots + a_k : \mathbf{a} \in \mathsf{Z}(n)\}$$
$$\mathsf{L}(n) = \{\ell_1 < \ldots < \ell_r\} \rightsquigarrow \Delta(n) = \{\ell_i - \ell_{i-1}\}$$

Compute $\Delta(S) = \bigcup_n \Delta(n)$.

Computing the delta set of a numerical monoid

Theorem (García-García–Moreno-Frías–Vigneron-Tenorio, 2014)

$S = \langle n_1, \ldots, n_k \rangle$. For $n \ge N_S$, $\Delta(n) = \Delta(n + \mathrm{lcm}(n_1 n_k))$.

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A solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$.

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Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \leq k$,

$$\begin{aligned} \phi_i : \mathsf{Z}(n - n_i) &\longrightarrow \mathsf{Z}(n) \\ \mathbf{a} &\longmapsto \mathbf{a} + \mathbf{e}_i \end{aligned}$$

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Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \leq k$,

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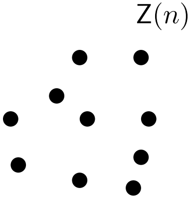

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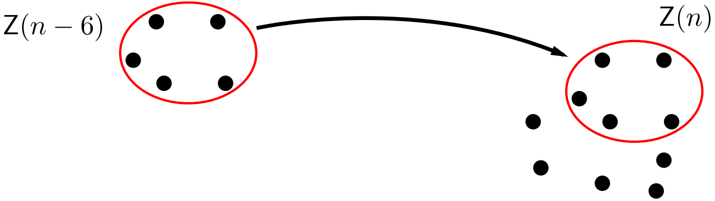

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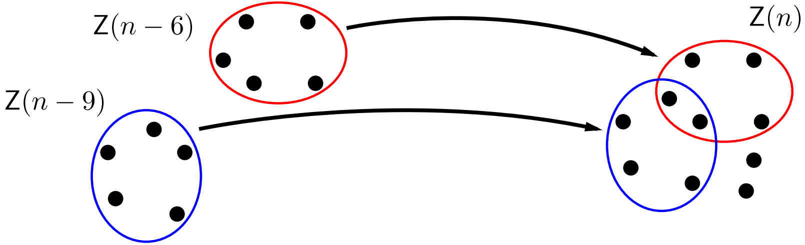

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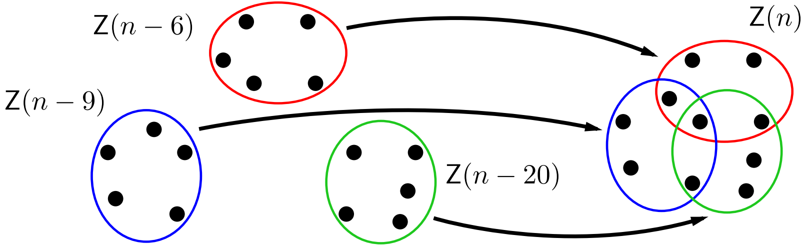

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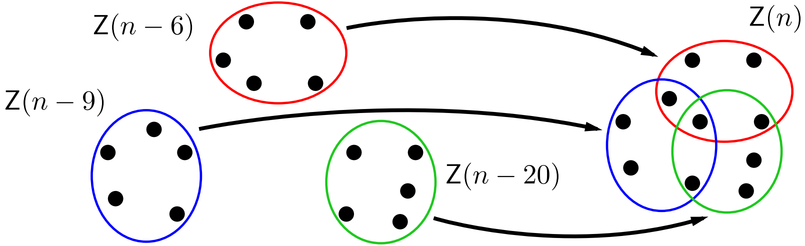

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$n \in S = \langle 6, 9, 20 \rangle$	$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0	$\{\mathbf{0}\}$	$\{0\}$

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$n \in S = \langle 6, 9, 20 \rangle$		$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0		$\{\mathbf{0}\}$	$\{0\}$
6	$\mathbf{0} \overset{6}{\rightsquigarrow} \mathbf{e}_1$	$\{\mathbf{e}_1\}$	$\{1\}$

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9	$\mathbf{0} \overset{9}{\rightsquigarrow} \mathbf{e}_2$	$\{\mathbf{e}_2\}$	$\{1\}$
12	$\mathbf{e}_1 \overset{6}{\rightsquigarrow} 2\mathbf{e}_1$	$\{2\mathbf{e}_1\}$	$\{2\}$

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18	$2\mathbf{e}_1 \overset{6}{\rightsquigarrow} 3\mathbf{e}_1$	$\{3\mathbf{e}_1, 2\mathbf{e}_2\}$	$\{2, 3\}$

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18	$2\mathbf{e}_1 \overset{6}{\rightsquigarrow} 3\mathbf{e}_1$	$\{3\mathbf{e}_1, 2\mathbf{e}_2\}$	$\{2, 3\}$
	$\mathbf{e}_2 \overset{9}{\rightsquigarrow} 2\mathbf{e}_2$		
20	$\mathbf{0} \overset{20}{\rightsquigarrow} \mathbf{e}_3$	$\{\mathbf{e}_3\}$	$\{1\}$

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20	$\mathbf{0} \overset{20}{\rightsquigarrow} \mathbf{e}_3$	$\{\mathbf{e}_3\}$	$\{1\}$
$\vdots$		$\vdots$	$\vdots$

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$$\mathbf{a} \longmapsto \mathbf{a} + \mathbf{e}_i$$

$$\mathsf{Z}(n) = \textstyle\bigcup_{i \leq k} \phi_i(\mathsf{Z}(n - n_i))$$

$n \in S = \langle 6, 9, 20 \rangle$		$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0		$\{\mathbf{0}\}$	$\{0\}$
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⋮		⋮	⋮

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Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \le k$,

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⋮		⋮	⋮

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$$\mathsf{Z}(n) = \bigcup_{i \leq k} \phi_i(\mathsf{Z}(n - n_i)) \qquad \mathsf{L}(n) = \bigcup_{i \leq k} \psi_i(\mathsf{L}(n - n_i))$$

$n \in S = \langle 6, 9, 20 \rangle$		$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0		$\{\mathbf{0}\}$	$\{0\}$
6	$\mathbf{0} \overset{6}{\rightsquigarrow} \mathbf{e}_1$	$\{\mathbf{e}_1\}$	$\{1\}$
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20	$\mathbf{0} \overset{20}{\rightsquigarrow} \mathbf{e}_3$	$\{\mathbf{e}_3\}$	$\{1\}$
$\vdots$		$\vdots$	$\vdots$

A solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \leq k$,

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$$\mathsf{Z}(n) = \bigcup_{i \leq k} \phi_i(\mathsf{Z}(n - n_i)) \qquad \mathsf{L}(n) = \bigcup_{i \leq k} \psi_i(\mathsf{L}(n - n_i))$$

$n \in S = \langle 6, 9, 20 \rangle$	$\mathsf{L}(n)$
0	$\{0\}$
6	
9	
12	
15	
18	
20	
$\vdots$	

A solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \le k$,

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$n \in S = \langle 6, 9, 20 \rangle$	$\mathsf{L}(n)$	
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6	$\{1\}$	$0 \overset{6}{\rightsquigarrow} 1$
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18	$\{2, 3\}$	$2 \overset{6}{\rightsquigarrow} 3$
20		
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Computing the delta set dynamically

Theorem (García-García–Moreno-Frías–Vigneron-Tenorio, 2014)

$S = \langle n_1, \ldots, n_k \rangle$. *For* $n \geq N_S$, $\Delta(n) = \Delta(n + \operatorname{lcm}(n_1, n_k))$.

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compute:

$$\mathsf{Z}(n) = \{\mathbf{a} \in \mathbb{N}^k : n = a_1 n_1 + \cdots + a_k n_k\}$$
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Compute $\Delta(S) = \bigcup_n \Delta(n)$.

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$$|\mathsf{Z}(n)| \quad \approx \quad n^{k-1}$$

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Runtime comparison

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S	N_S	$\Delta(S)$	Existing	Dynamic
$\langle 7, 15, 17, 18, 20\rangle$	1935	$\{1, 2, 3\}$	1m 28s	146ms
$\langle 11, 53, 73, 87\rangle$	14381	$\{2, 4, 6, 8, 10, 22\}$	0m 49s	2.5s
$\langle 31, 73, 77, 87, 91\rangle$	31364	$\{2, 4, 6\}$	400m 12s	4.2s
$\langle 100, 121, 142, 163, 284\rangle$	24850	$\{21\}$	———	0m 3.6s
$\langle 1001, 1211, 1421, 1631, 2841\rangle$	2063141	$\{10, 20, 30\}$	———	1m 56s

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GAP Numerical Semigroups Package, available at

`http://www.gap-system.org/Packages/numericalsgps.html`.

ω-primality

Definition (ω-primality)

Fix a (multiplicatively written) monoid $(M, \cdot)$. For $x \in M$, $\omega(x)$ is the smallest positive integer m such that whenever $x \mid \prod_{i=1}^r u_i$ for $r > m$, there exists a subset $T \subset \{1, \ldots, r\}$ with $|T| \leq m$ such that $x \mid \prod_{i \in T} u_i$.

Fact

$\omega(x) = 1$ if and only if x is prime (i.e. $x \mid ab$ implies $x \mid a$ or $x \mid b$).

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Fact

M is factorial if and only if every irreducible element $u \in M$ is prime. Moreover, $\omega(p_1 \cdots p_r) = r$ for any primes $p_1, \ldots, p_r \in M$.

ω-primality

Definition (ω-primality)

Fix a numerical monoid S and $n \in S$.
$\omega_S(n)$ is the minimal m such that whenever $(\sum_{i=1}^r x_i) - n \in S$ for $r > m$, there exists $T \subset \{1, \ldots, r\}$ with $|T| \leq m$ and $(\sum_{i \in T} x_i) - n \in S$.

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Definition

A *bullet* for $n \in S$ is a tuple $\mathbf{b} = (b_1, \ldots, b_k) \in \mathbb{N}^k$ such that

(i) $b_1 n_1 + \cdots + b_k n_k - n \in S$, and

(ii) $b_1 n_1 + \cdots + (b_i - 1) n_i + \cdots + b_k n_k - n \notin S$ for each $b_i > 0$.

The set of bullets of n is denoted $\mathrm{bul}(n)$.

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Proposition

$\omega_S(n) = \max\{|\mathbf{b}| : \mathbf{b} \in \mathrm{bul}(n)\}$.

Quasilinearity for ω-primality

Theorem ((O.–Pelayo, 2013), (García-García et.al., 2013))

$\omega_S(n) = \frac{1}{n_1}n + a_0(n)$ for $n \gg 0$, where $a_0(n)$ periodic with period n_1.

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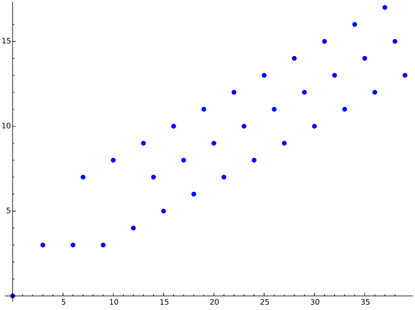
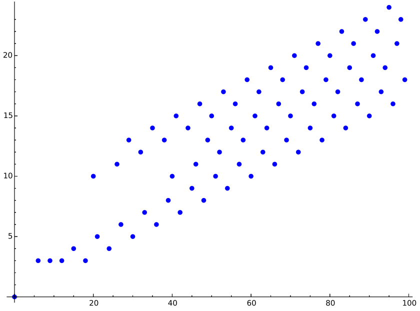


$S = \langle 3, 7 \rangle$



$S = \langle 6, 9, 20 \rangle$

Using bullets to compute ω-primality

Algorithm: Compute $\mathrm{bul}(n)$, then compute $\omega(n) = \max\{|\mathbf{b}| : \mathbf{b} \in \mathrm{bul}(n)\}$.

Using bullets to compute ω-primality

Algorithm: Compute $\text{bul}(n)$, then compute $\omega(n) = \max\{|\mathbf{b}| : \mathbf{b} \in \text{bul}(n)\}$.

Example

$S = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$.

Using bullets to compute ω-primality

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Example

$S = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$.

$$\text{bul}(20) = \{(4,4,0), (7,2,0), (10,0,0), (1,6,0), (0,8,0), (0,0,1)\}$$

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$S = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$.

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$$8 \cdot 9 - 20 = 52 \in S$$

Using bullets to compute ω-primality

Algorithm: Compute $\text{bul}(n)$, then compute $\omega(n) = \max\{|\mathbf{b}| : \mathbf{b} \in \text{bul}(n)\}$.

Example

$S = \langle 6, 9, 20\rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$.

$$\text{bul}(20) = \{(4,4,0),(7,2,0),(10,0,0),(1,6,0),(0,8,0),(0,0,1)\}$$

$$8 \cdot 9 - 20 = 52 \in S$$
$$7 \cdot 9 - 20 = 43 \notin S$$

Using bullets to compute ω-primality

Algorithm: Compute $\mathrm{bul}(n)$, then compute $\omega(n) = \max\{|\mathbf{b}| : \mathbf{b} \in \mathrm{bul}(n)\}$.

Example

$S = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$.

$$\mathrm{bul}(20) = \{(4,4,0), (7,2,0), (10,0,0), (1,6,0), (0,8,0), (0,0,1)\}$$

$$\begin{array}{l} 8 \cdot 9 - 20 = 52 \in S \\ 7 \cdot 9 - 20 = 43 \notin S \end{array} \quad \Rightarrow \quad (0,8,0) \in \mathrm{bul}(20)$$

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Algorithm: Compute $\text{bul}(n)$, then compute $\omega(n) = \max\{|\mathbf{b}| : \mathbf{b} \in \text{bul}(n)\}$.

Example

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$$\text{bul}(20) = \{(4,4,0), (7,2,0), (10,0,0), (1,6,0), (0,8,0), (0,0,1)\}$$

$$\begin{array}{l} 8 \cdot 9 - 20 = 52 \in S \\ 7 \cdot 9 - 20 = 43 \notin S \end{array} \quad \Rightarrow \quad (0,8,0) \in \text{bul}(20)$$

$$1 \cdot 6 + 6 \cdot 9 - 20 = 40 \in S$$

Using bullets to compute ω-primality

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Example

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$$\text{bul}(20) = \{(4,4,0), (7,2,0), (10,0,0), (1,6,0), (0,8,0), (0,0,1)\}$$

$$\begin{array}{l} 8 \cdot 9 - 20 = 52 \in S \\ 7 \cdot 9 - 20 = 43 \notin S \end{array} \quad \Rightarrow \quad (0,8,0) \in \text{bul}(20)$$

$$\begin{aligned} 1 \cdot 6 + 6 \cdot 9 - 20 &= 40 \in S \\ 6 \cdot 9 - 20 &= 34 \notin S \end{aligned}$$

Using bullets to compute ω-primality

Algorithm: Compute $\mathrm{bul}(n)$, then compute $\omega(n) = \max\{|\mathbf{b}| : \mathbf{b} \in \mathrm{bul}(n)\}$.

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Moral of (the remainder of) this talk: bullets behave like factorizations!

Toward a dynamic algorithm... the inductive step

Recall: for $n \in S = \langle n_1, \ldots n_k \rangle$, $\mathsf{Z}(n) = \{\mathbf{a} \in \mathbb{N}^k : \sum_{i=1}^{k} a_i n_i = n\}$.

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Definition/Proposition (Cover morphisms)

Fix $n \in S$ and $i \leq k$. The *i-th cover morphism for n* is the map

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$$\mathbf{b} \longmapsto \begin{cases} \mathbf{b} + \mathbf{e}_i & \sum_{j=1}^k b_j n_j - n - n_i \notin S \\ \mathbf{b} & \sum_{j=1}^k b_j n_j - n - n_i \in S \end{cases}$$

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Moreover, $\mathsf{bul}(n) = \bigcup_{i \leq k} \psi_i(\mathsf{bul}(n - n_i))$.**

Toward a dynamic algorithm. . . the base case

Definition (ω-primality in numerical monoids)

Fix a numerical monoid S and $n \in S$.

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$\omega_S(n)$ is the minimal m such that whenever $(\sum_{i=1}^{r} x_i) - n \in S$ for $r > m$, there exists $T \subset \{1, \ldots, r\}$ with $|T| \le m$ and $(\sum_{i \in T} x_i) - n \in S$.

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Remark

All properties of ω extend from S to $\mathbb{Z}$.

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Remark

All properties of ω extend from S to $\mathbb{Z}$.

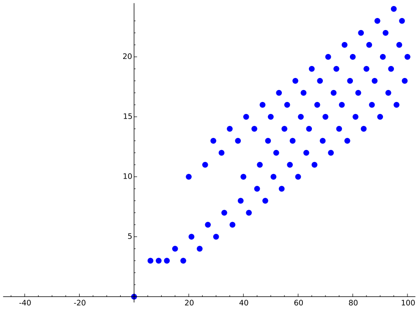
Proposition

For $n \in \mathbb{Z}$, the following are equivalent:

(i) $\omega(n) = 0$,
(ii) $\mathsf{bul}(n) = \{\mathbf{0}\}$,
(iii) $-n \in S$.

A dynamic algorithm!

ω-primality for $n \in S$:



$$S = \langle 6, 9, 20 \rangle$$

A dynamic algorithm!

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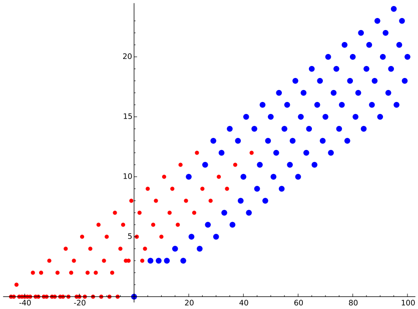

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An action shot

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≤ -44	0	$\{\mathbf{0}\}$

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-32	0	$\{\mathbf{0}\}$
-31	3	$\{3\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

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6	3	$\{3\mathbf{e}_3, 2\mathbf{e}_2, \ldots\}$

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6	3	$\{3\mathbf{e}_3, 2\mathbf{e}_2, \ldots\}$
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8	8	$\{8\mathbf{e}_1, (5, 2, 0), \ldots\}$
9	3	$\{3\mathbf{e}_1, 3\mathbf{e}_3, \ldots\}$
10	5	$\{5\mathbf{e}_1, (2, 2, 0), \ldots\}$
11	10	$\{10\mathbf{e}_1, (4, 3, 0), \ldots\}$
12	3	$\{3\mathbf{e}_3, 2\mathbf{e}_1, \ldots\}$
13	7	$\{7\mathbf{e}_1, (1, 3, 0), \ldots\}$
14	9	$\{9\mathbf{e}_1, (6, 2, 0), \ldots\}$
15	4	$\{4\mathbf{e}_1, (6, 2, 0), \ldots\}$
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11	10	$\{10\mathbf{e}_1, (4, 3, 0), \ldots\}$
12	3	$\{3\mathbf{e}_3, 2\mathbf{e}_1, \ldots\}$
13	7	$\{7\mathbf{e}_1, (1, 3, 0), \ldots\}$
14	9	$\{9\mathbf{e}_1, (6, 2, 0), \ldots\}$
15	4	$\{4\mathbf{e}_1, (6, 2, 0), \ldots\}$
$\vdots$	$\vdots$	$\vdots$
149	33	$\{33\mathbf{e}_1, \ldots\}$

An action shot

$S = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$.

$n \in \mathbb{Z}$	$\omega(n)$	$\mathsf{bul}(n)$
≤ -44	0	$\{\mathbf{0}\}$
-43	1	$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$
-42	0	$\{\mathbf{0}\}$
$\vdots$	$\vdots$	$\vdots$
-38	0	$\{\mathbf{0}\}$
-37	2	$\{2\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$
-36	0	$\{\mathbf{0}\}$
-35	0	$\{\mathbf{0}\}$
-34	2	$\{\mathbf{e}_1, 2\mathbf{e}_2, \mathbf{e}_3\}$
-33	0	$\{\mathbf{0}\}$
-32	0	$\{\mathbf{0}\}$
-31	3	$\{3\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$
$\vdots$	$\vdots$	$\vdots$

$n \in \mathbb{Z}$	$\omega(n)$	$\mathsf{bul}(n)$
6	3	$\{3\mathbf{e}_3, 2\mathbf{e}_2, \ldots\}$
7	6	$\{6\mathbf{e}_1, (3, 1, 0), \ldots\}$
8	8	$\{8\mathbf{e}_1, (5, 2, 0), \ldots\}$
9	3	$\{3\mathbf{e}_1, 3\mathbf{e}_3, \ldots\}$
10	5	$\{5\mathbf{e}_1, (2, 2, 0), \ldots\}$
11	10	$\{10\mathbf{e}_1, (4, 3, 0), \ldots\}$
12	3	$\{3\mathbf{e}_3, 2\mathbf{e}_1, \ldots\}$
13	7	$\{7\mathbf{e}_1, (1, 3, 0), \ldots\}$
14	9	$\{9\mathbf{e}_1, (6, 2, 0), \ldots\}$
15	4	$\{4\mathbf{e}_1, (6, 2, 0), \ldots\}$
$\vdots$	$\vdots$	$\vdots$
149	33	$\{33\mathbf{e}_1, \ldots\}$
150	25	$\{25\mathbf{e}_1, \ldots\}$

An action shot

$S = \langle 6, 9, 20 \rangle = \{0, 6, 9, 12, 15, 18, 20, 21, \ldots\}$.

$n \in \mathbb{Z}$	$\omega(n)$	$\text{bul}(n)$

$n \in \mathbb{Z}$	$\omega(n)$	$\text{bul}(n)$
6	3	$\{3\mathbf{e}_3, 2\mathbf{e}_2, \ldots\}$
9	3	$\{3\mathbf{e}_1, 3\mathbf{e}_3, \ldots\}$
12	3	$\{3\mathbf{e}_3, 2\mathbf{e}_1, \ldots\}$
15	4	$\{4\mathbf{e}_1, (6, 2, 0), \ldots\}$
$\vdots$	$\vdots$	$\vdots$
149	33	$\{33\mathbf{e}_1, \ldots\}$
150	25	$\{25\mathbf{e}_1, \ldots\}$

Runtime comparison

Runtime comparison

S	$n \in S$	$\omega_S(n)$	Existing	Dynamic
$\langle 6, 9, 20 \rangle$	1000	170	1m 1.3s	6ms
$\langle 11, 13, 15 \rangle$	1000	97	0m 10.7s	5ms
$\langle 11, 13, 15 \rangle$	3000	279	14m 34.7s	15ms
$\langle 11, 13, 15 \rangle$	10000	915	———	42ms
$\langle 15, 27, 32, 35 \rangle$	1000	69	3m 54.7s	9ms
$\langle 100, 121, 142, 163, 284 \rangle$	25715	308	———	0m 27s
$\langle 1001, 1211, 1421, 1631, 2841 \rangle$	357362	405	———	57m 27s

GAP Numerical Semigroups Package, available at

`http://www.gap-system.org/Packages/numericalsgps.html`.

Other dynamic directions

Other dynamic directions

Fact

Dynamic algorithms rock.

Other dynamic directions

Fact

Dynamic algorithms rock.

Problem

What about catenary degree?

Other dynamic directions

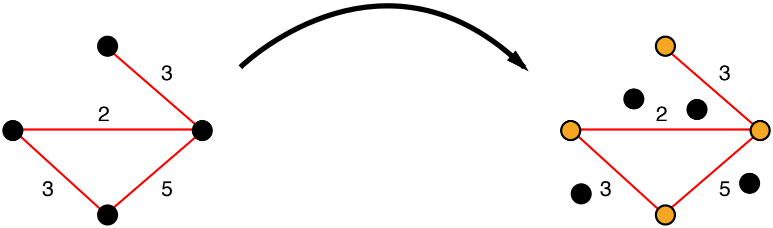
Fact

Dynamic algorithms rock.

Problem

What about catenary degree?

Cover morphisms:



$$Z_S(n) \hookrightarrow Z_S(n + n_i)$$

$$\mathbf{a} \longmapsto \mathbf{a} + \mathbf{e}_i$$

References

C. O'Neill, R. Pelayo (2014)
How do you measure primality?
American Mathematical Monthly, **122** (2014), no. 2, 121–137.

J. García-García, M. Moreno-Frías, A. Vigneron-Tenorio (2014)
Computation of delta sets of numerical monoids.
preprint.

T. Barron, C. O'Neill, R. Pelayo (2015)
On the computation of delta sets and ω-primality in numerical monoids.
preprint.

M. Delgado, P. García-Sánchez, J. Morais
GAP Numerical Semigroups Package
`http://www.gap-system.org/Packages/numericalsgps.html`.

References

C. O'Neill, R. Pelayo (2014)
How do you measure primality?
American Mathematical Monthly, **122** (2014), no. 2, 121–137.

J. García-García, M. Moreno-Frías, A. Vigneron-Tenorio (2014)
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Thanks!