Invariants of non-unique factorization

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Factorial domains

Definition

An integral domain R is *factorial* if for each non-unit $r \in R$,

1. there is a *factorization* $r = u_1 \cdots u_k$ as a product of irreducibles, and
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The point: it's nontrivial.

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$$\begin{array}{rcl} (R, +, \cdot) & \rightsquigarrow & (R \setminus \{0\}, \cdot) \\ (\mathbb{C}[M], +, \cdot) & \rightsquigarrow & (M, \cdot) \end{array}$$

Interesting monoids

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An *arithmetical congruence monoid* is a **multiplicative** submonoid

$$M_{a,b} = \{n : n \equiv a \bmod b\} \subset \mathbb{Z}_{>0}$$

for $a, b > 0$ with $a^2 \equiv a \bmod b$.

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- $$\begin{aligned} 441 &= 9 \cdot 49 = 21 \cdot 21 \\ &= (3^2) \cdot (7^2) = (3 \cdot 7) \cdot (3 \cdot 7). \end{aligned}$$

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Fix a commutative, cancellative monoid $(M, \cdot)$. For each non-unit $m \in M$,

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- This is (almost) the best we could hope for.

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- $\rho(n) \leq 20/6$ for all $n \in McN$.

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- Elasticity of $M_{4,6}$ is *not accepted*.

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Let $S = \langle n_1, \ldots, n_k \rangle \subset (\mathbb{N}, +)$. For $n \in S$,

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Theorem (Barron–O–Pelayo, 2014)

Let $S = \langle n_1, \ldots, n_k \rangle$. For $n > n_k(n_{k-1} - 1)$,

$$\begin{aligned} \mathsf{M}(n + n_1) &= 1 + \mathsf{M}(n) \\ \mathsf{m}(n + n_k) &= 1 + \mathsf{m}(n) \end{aligned}$$

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Equivalently, $\mathsf{M}(n)$, $\mathsf{m}(n)$ *are eventually quasilinear:*

$$\begin{aligned} \mathsf{M}(n) &= \tfrac{1}{n_1} n + a_0(n) \\ \mathsf{m}(n) &= \tfrac{1}{n_k} n + b_0(n) \end{aligned}$$

for periodic functions $a_0(n)$, $b_0(n)$.

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$n = 1000001$.

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$S = \langle 6, 9, 20 \rangle$:

$n = 1000001$.

$$\mathrm{m}(1000001) \quad = \quad ?$$

Factorization lengths in numerical monoids

Theorem (Barron–O–Pelayo, 2014)

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$$\begin{aligned}
\mathsf{m}(1000001) &= ? \\
\mathsf{m}(181) &= 11 \quad \text{and} \quad 181 = 2(6) + 1(9) + 8(20) \\
\mathsf{m}(1000001) &= 50002 \quad \text{and} \quad 1000001 = 2(6) + 1(9) + 49993(20)
\end{aligned}$$

Factorization lengths in numerical monoids

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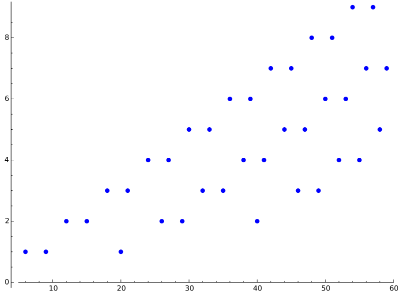
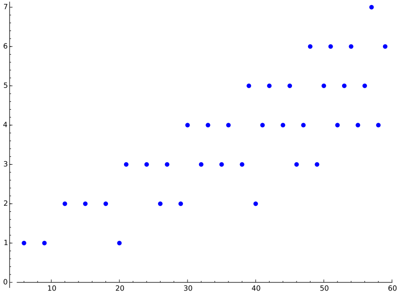


$\mathsf{M}(n) : S \to \mathbb{N}$



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$S = \langle 5, 16, 17, 18, 19 \rangle$:

Factorization lengths in numerical monoids

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Let $S = \langle n_1, \ldots, n_k \rangle$. For $n > n_k(n_{k-1} - 1)$,

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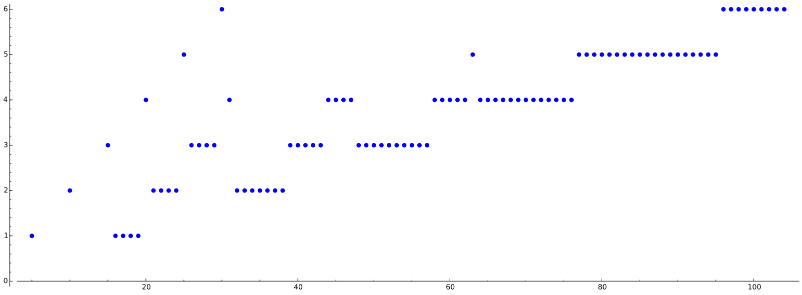

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