Computing the delta set of a numerical monoid

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Joint with Thomas Barron and Roberto Pelayo

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Theorem (Chapman–Hoyer–Kaplan, 2000)

$S = \langle n_1, \ldots, n_k \rangle$. *For* $n \geq 2kn_2n_k^2$, $\Delta(n) = \Delta(n + n_1n_k)$.

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Theorem (García-García–Moreno-Frías–Vigneron-Tenorio, 2014)

$S = \langle n_1, \ldots, n_k \rangle$. *For* $n \geq N_S$, $\Delta(n) = \Delta(n + \mathrm{lcm}(n_1, n_k))$.

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$$|\mathsf{Z}(n)| \approx n^{k-1}$$

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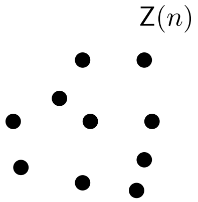

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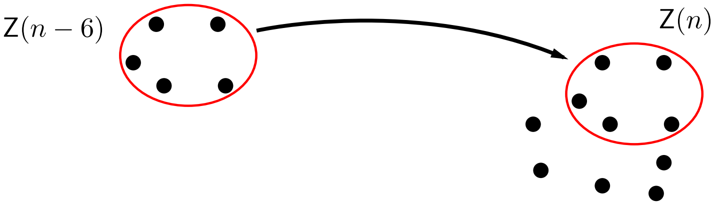

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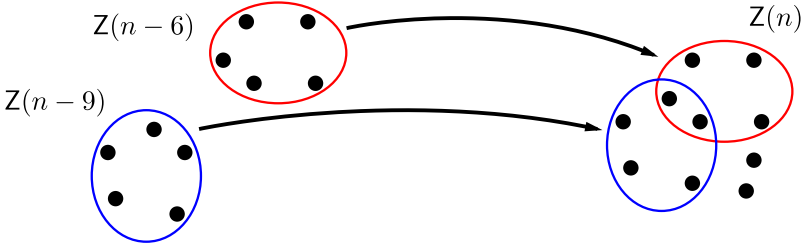

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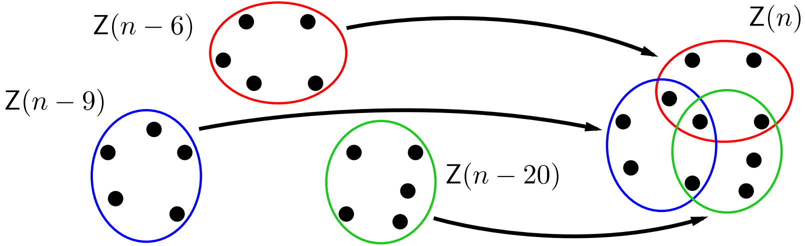

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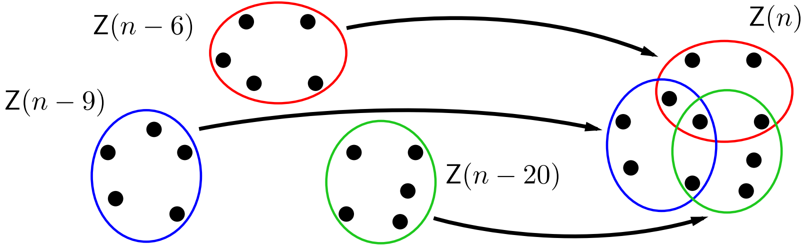

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12	$\mathbf{e}_1 \overset{6}{\rightsquigarrow} 2\mathbf{e}_1$	$\{2\mathbf{e}_1\}$	$\{2\}$
15	$\mathbf{e}_2 \overset{6}{\rightsquigarrow} (1, 1, 0)$	$\{(1, 1, 0)\}$	$\{2\}$

A solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \leq k$,

$$\begin{array}{rcl} \phi_i : \mathsf{Z}(n - n_i) & \longrightarrow & \mathsf{Z}(n) \\ \mathbf{a} & \longmapsto & \mathbf{a} + \mathbf{e}_i \end{array}$$

$$\mathsf{Z}(n) = \bigcup_{i \leq k} \phi_i(\mathsf{Z}(n - n_i))$$

$n \in McN = \langle 6, 9, 20 \rangle$		$\mathsf{Z}(n)$	$\mathsf{L}(n)$
0		$\{\mathbf{0}\}$	$\{0\}$
6	$\mathbf{0} \overset{6}{\rightsquigarrow} \mathbf{e}_1$	$\{\mathbf{e}_1\}$	$\{1\}$
9	$\mathbf{0} \overset{9}{\rightsquigarrow} \mathbf{e}_2$	$\{\mathbf{e}_2\}$	$\{1\}$
12	$\mathbf{e}_1 \overset{6}{\rightsquigarrow} 2\mathbf{e}_1$	$\{2\mathbf{e}_1\}$	$\{2\}$
15	$\mathbf{e}_2 \overset{6}{\rightsquigarrow} (1,1,0)$	$\{(1,1,0)\}$	$\{2\}$
	$\mathbf{e}_1 \overset{9}{\rightsquigarrow} (1,1,0)$		

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Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \leq k$,

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0		$\{\mathbf{0}\}$	$\{0\}$
6	$\mathbf{0} \overset{6}{\rightsquigarrow} \mathbf{e}_1$	$\{\mathbf{e}_1\}$	$\{1\}$
9	$\mathbf{0} \overset{9}{\rightsquigarrow} \mathbf{e}_2$	$\{\mathbf{e}_2\}$	$\{1\}$
12	$\mathbf{e}_1 \overset{6}{\rightsquigarrow} 2\mathbf{e}_1$	$\{2\mathbf{e}_1\}$	$\{2\}$
15	$\mathbf{e}_2 \overset{6}{\rightsquigarrow} (1,1,0)$	$\{(1,1,0)\}$	$\{2\}$
	$\mathbf{e}_1 \overset{9}{\rightsquigarrow} (1,1,0)$		
18	$2\mathbf{e}_1 \overset{6}{\rightsquigarrow} 3\mathbf{e}_1$	$\{3\mathbf{e}_1, 2\mathbf{e}_2\}$	$\{2, 3\}$

A solution: dynamic programming

Fix $n \in S = \langle n_1, \ldots, n_k \rangle$. For each $i \leq k$,

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12	$\mathbf{e}_1 \overset{6}{\rightsquigarrow} 2\mathbf{e}_1$	$\{2\mathbf{e}_1\}$	$\{2\}$
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A solution: dynamic programming

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18	$2\mathbf{e}_1 \overset{6}{\rightsquigarrow} 3\mathbf{e}_1$	$\{3\mathbf{e}_1, 2\mathbf{e}_2\}$	$\{2, 3\}$
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20	$\mathbf{0} \overset{20}{\rightsquigarrow} \mathbf{e}_3$	$\{\mathbf{e}_3\}$	$\{1\}$
$\vdots$		$\vdots$	$\vdots$

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$\vdots$		$\vdots$	$\vdots$

A solution: dynamic programming

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$$\mathsf{Z}(n) = \bigcup_{i \leq k} \phi_i(\mathsf{Z}(n - n_i))$$

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0		$\{\mathbf{0}\}$	$\{0\}$
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$n \in McN = \langle 6, 9, 20 \rangle$	$\mathsf{L}(n)$
0	$\{0\}$
6	
9	
12	
15	
18	
20	
$\vdots$	

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$n \in McN = \langle 6, 9, 20 \rangle$	$\mathsf{L}(n)$	
0	$\{0\}$	
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15		
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20		
⋮		

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18	$\{2, 3\}$	$2 \overset{6}{\rightsquigarrow} 3$
20		
$\vdots$		

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$\vdots$	$\vdots$	$\vdots$

Computing the delta set dynamically

Theorem (García-García–Moreno-Frías–Vigneron-Tenorio, 2014)

$S = \langle n_1, \ldots, n_k \rangle$. *For* $n \geq N_S$, $\Delta(n) = \Delta(n + \operatorname{lcm}(n_1, n_k))$.

Computing the delta set dynamically

Theorem (García-García–Moreno-Frías–Vigneron-Tenorio, 2014)

$S = \langle n_1, \ldots, n_k \rangle$. *For* $n \geq N_S$, $\Delta(n) = \Delta(n + \mathrm{lcm}(n_1, n_k))$.

For $n \in S$ with $0 \leq n \leq N_S + \mathrm{lcm}(n_1, n_k)$, compute:

$$\mathsf{Z}(n) = \{\mathbf{a} \in \mathbb{N}^k : n = a_1 n_1 + \cdots + a_k n_k\}$$
$$\mathsf{Z}(n) \rightsquigarrow \mathsf{L}(n)$$
$$\mathsf{L}(n) \rightsquigarrow \Delta(n)$$

Compute $\Delta(S) = \bigcup_n \Delta(n)$.

Computing the delta set dynamically

Theorem (García-García–Moreno-Frías–Vigneron-Tenorio, 2014)

$S = \langle n_1, \ldots, n_k \rangle$. *For* $n \geq N_S$, $\Delta(n) = \Delta(n + \mathrm{lcm}(n_1, n_k))$.

For $n \in S$ with $0 \leq n \leq N_S + \mathrm{lcm}(n_1, n_k)$,
compute:

$$\sout{\mathsf{Z}(n) = \{\mathbf{a} \in \mathbb{N}^k : n = a_1 n_1 + \cdots + a_k n_k\}}$$

$$\mathsf{L}(n - *) \rightsquigarrow \mathsf{L}(n)$$

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S	N_S	$\Delta(S)$	Existing	Dynamic
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$\langle 11, 53, 73, 87\rangle$	14381	$\{2, 4, 6, 8, 10, 22\}$	0m 49s	2.5s
$\langle 31, 73, 77, 87, 91\rangle$	31364	$\{2, 4, 6\}$	400m 12s	4.2s
$\langle 100, 121, 142, 163, 284\rangle$	24850	$\{21\}$	———	0m 3.6s
$\langle 1001, 1211, 1421, 1631, 2841\rangle$	2063141	$\{10, 20, 30\}$	———	1m 56s

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GAP Numerical Semigroups Package, available at

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