Minimal presentations of shifted numerical monoids

Christopher O'Neill

Texas A&M University

coneill@math.tamu.edu

Joint with Rebecca Conaway*, Felix Gotti, Jesse Horton*,
Roberto Pelayo, Mesa Williams*, and Brian Wissman

* = undergraduate student

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A *numerical monoid* S is an **additive** submonoid of $\mathbb{N}$ with $|\mathbb{N} \setminus S| < \infty$.

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$$\begin{array}{rcllcl} 60 & = & 7(6) + 2(9) & & \rightsquigarrow & (7, 2, 0) \\ & = & & 3(20) & \rightsquigarrow & (0, 0, 3) \end{array}$$

To shift a numerical monoid...

Fix $S = \langle r_1, \ldots, r_k \rangle \subset (\mathbb{N}, +)$, and let

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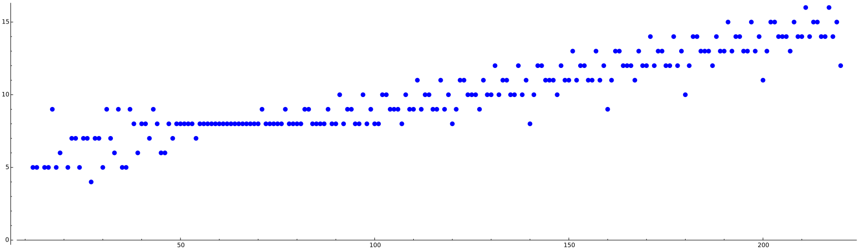

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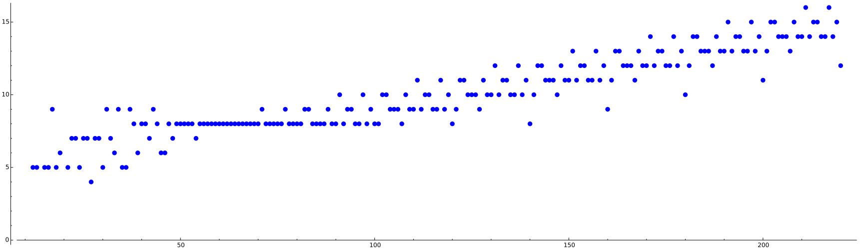

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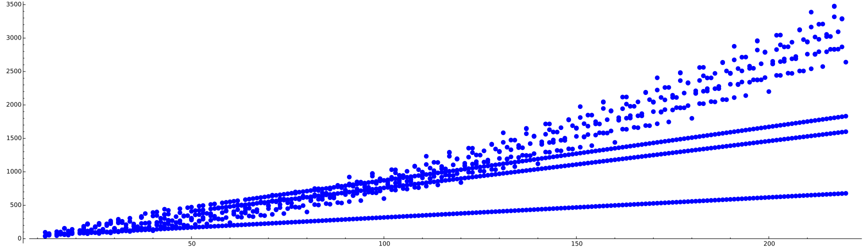
$M_n = \langle n, n + 6, n + 9, n + 20 \rangle$: Graded degrees for $\beta_0(M_n)$



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Underlying cause: minimal presentations!

Kernel congruences and minimal presentations

Let $S = \langle r_1, \ldots, r_k \rangle$.

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that is closed under *translation*.

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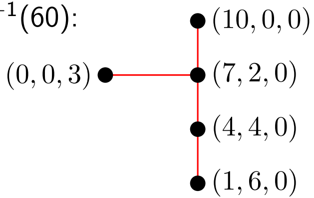

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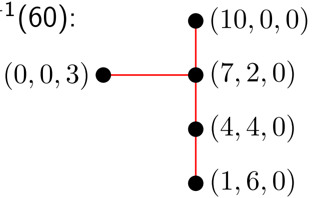

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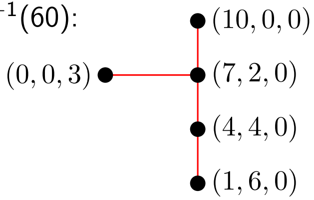

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$\mathrm{Cong}(\rho) = \ker \pi$ when the graph on $\pi^{-1}(n)$ is connected for all $n \in S$.

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M_{450}:

$$\left\{ \begin{array}{l} ((\ 0,0,8,0),(3,2,0,\ 3)),((\ 0,1,6,0),(4,0,0,\ 3)),((\ 0,3,0,0),(1,0,2,\ 0)), \\ ((20,5,0,0),(0,0,0,24)),((25,1,0,0),(0,0,4,21)),((26,0,0,0),(0,2,2,21)) \end{array} \right\}$$

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where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

Sneak peek for $M_n = \langle n, n + 6, n + 9, n + 20 \rangle$ and $n \gg 0$:

M_{450}:

$$\left\{ \begin{array}{l} ((\ 0,0,8,0),(3,2,0,\ 3)),((\ 0,1,6,0),(4,0,0,\ 3)),((\ 0,3,0,0),(1,0,2,\ 0)), \\ ((20,5,0,0),(0,0,0,24)),((25,1,0,0),(0,0,4,21)),((26,0,0,0),(0,2,2,21)) \end{array} \right\}$$

M_{470}:

$$\left\{ \begin{array}{l} ((\ 0,0,8,0),(3,2,0,\ 3)),((\ 0,1,6,0),(4,0,0,\ 3)),((\ 0,3,0,0),(1,0,2,\ 0)), \\ ((21,5,0,0),(0,0,0,25)),((26,1,0,0),(0,0,4,22)),((27,0,0,0),(0,2,2,22)) \end{array} \right\}$$

The shifting map

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M_{490}:

$$\left\{ \begin{array}{l} ((\ \ 0, 0, 8, 0), (3, 2, 0, \ \ 3)), ((\ \ 0, 1, 6, 0), (4, 0, 0, \ \ 3)), ((\ \ 0, 3, 0, 0), (1, 0, 2, \ \ 0)), \\ ((22, 5, 0, 0), (0, 0, 0, 26)), ((27, 1, 0, 0), (0, 0, 4, 23)), ((28, 0, 0, 0), (0, 2, 2, 23)) \end{array} \right\}$$

The shifting map

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- Φ_n is well-defined.

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- Φ_n is well-defined.

$$\pi_n(\mathbf{a}) = a_0 n + \textstyle\sum_{i=1}^k a_i(n + r_i) = |\mathbf{a}|n + \sum_{i=1}^k a_i r_i$$
$$\pi_{n+r_k}(\mathbf{a}) = \qquad\qquad = |\mathbf{a}|n + |\mathbf{a}|r_k + \textstyle\sum_{i=1}^k a_i r_i$$

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- Φ_n preserves reflexive and symmetric closure.

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- Φ_n preserves reflexive and symmetric closure.
- Φ_n preserves translation closure.

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$$\Phi_n((\mathbf{a}, \mathbf{a}') + (\mathbf{b}, \mathbf{b})) = \Phi_n(\mathbf{a}, \mathbf{a}') + (\mathbf{b}, \mathbf{b})$$

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- Only missing link: transitivity.

Monotone chains

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Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

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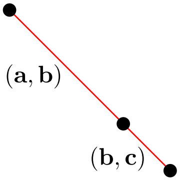

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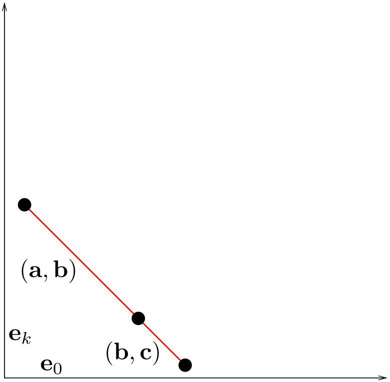

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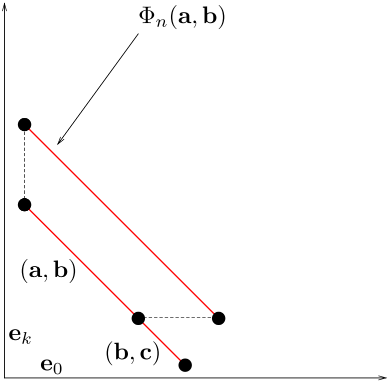

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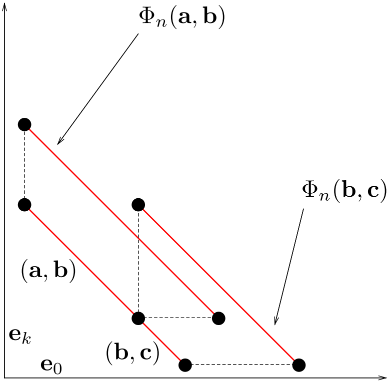

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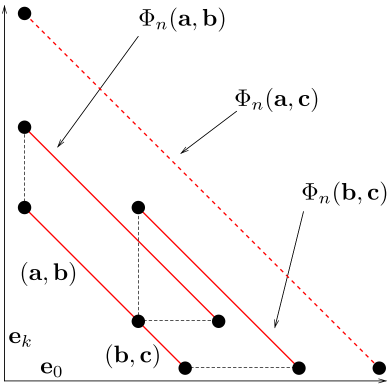

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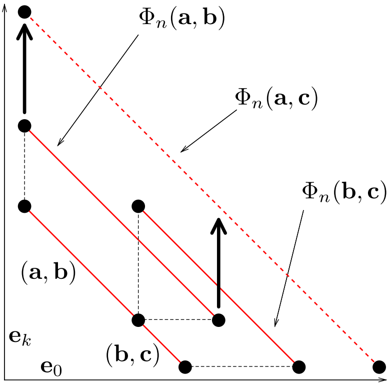

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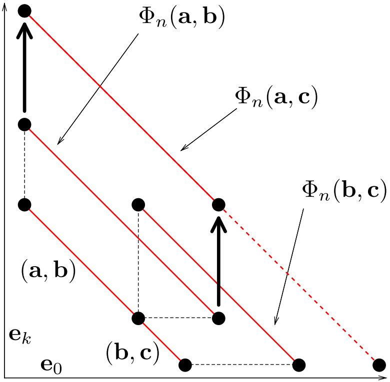

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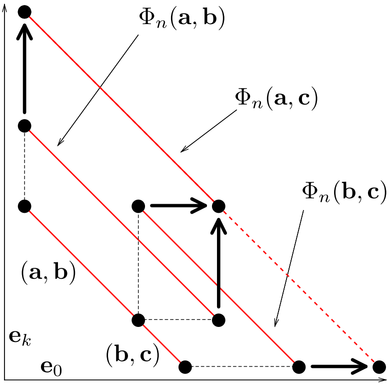

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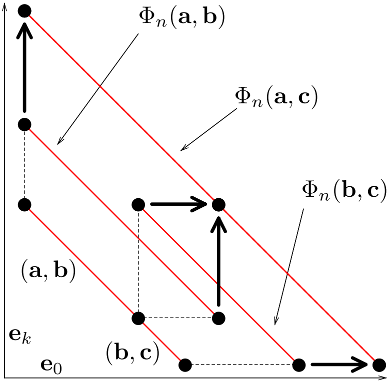

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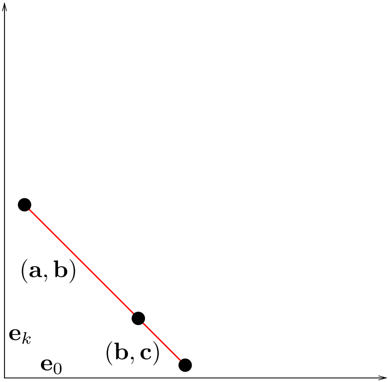

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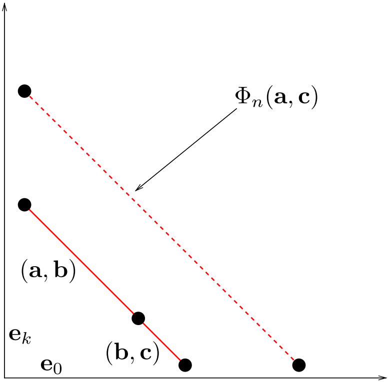

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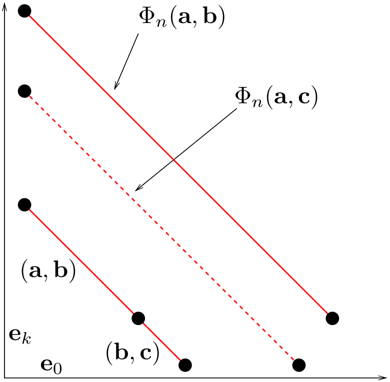

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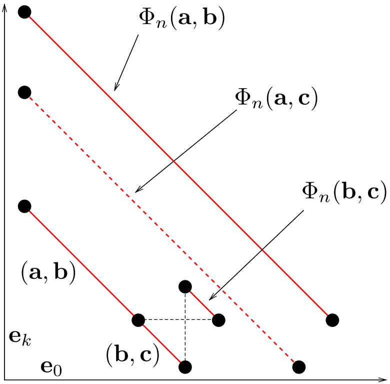

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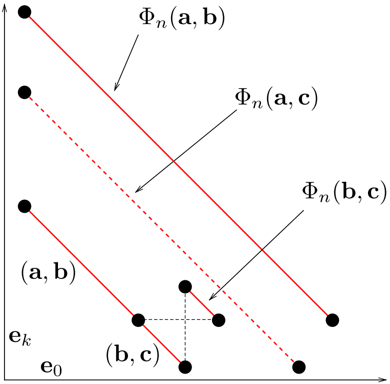
Need: *monotone* chains are sufficient for transitive closure.



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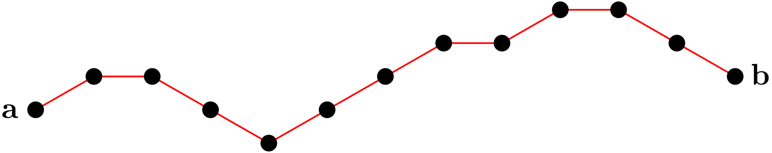

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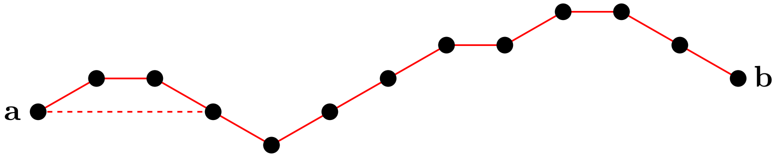

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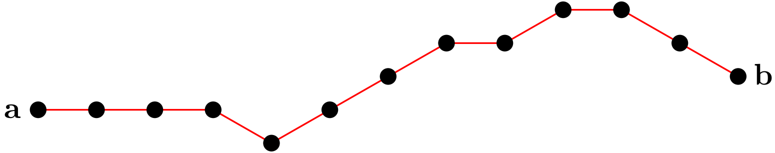

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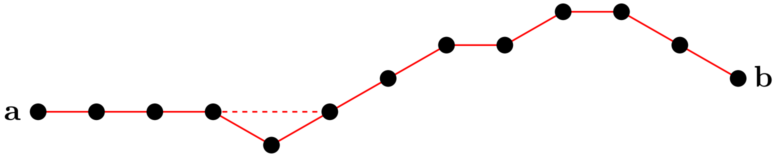

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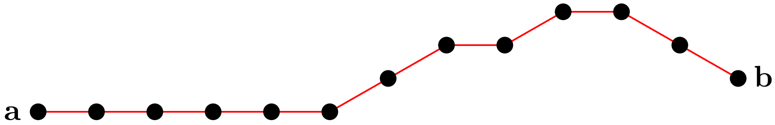

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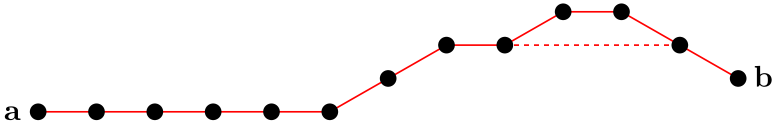

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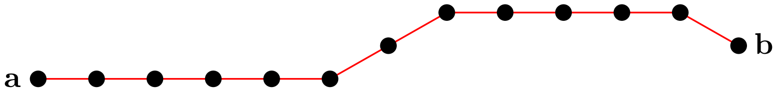

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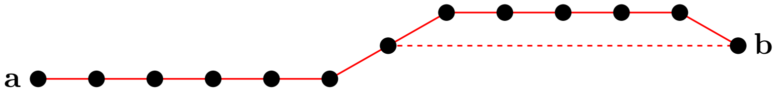

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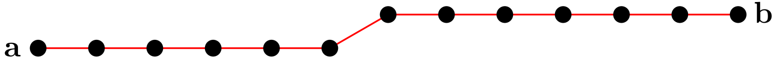

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Thanks!