Minimal presentations of shifted numerical monoids

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Possible factorization lengths for $n = 60$: $3, 7, 8, 9, 10$.

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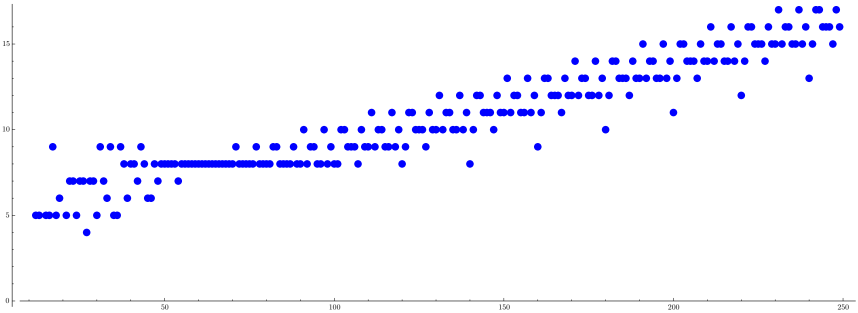

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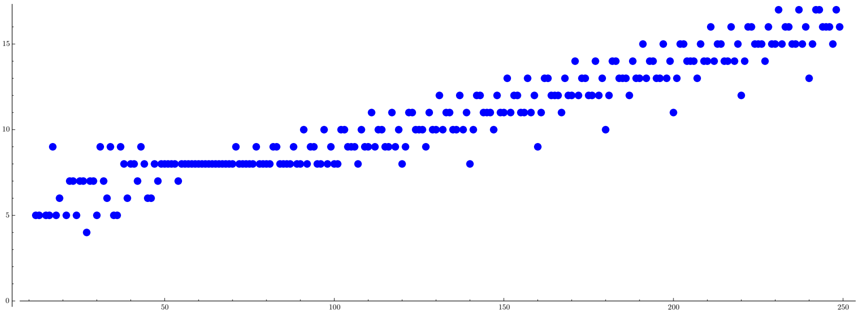
$c(M_n)$ is periodic-linear (quasilinear) for $n \geq 126$.



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$$\Delta(M_n) = \{1\} \text{ for all } n \geq 48$$

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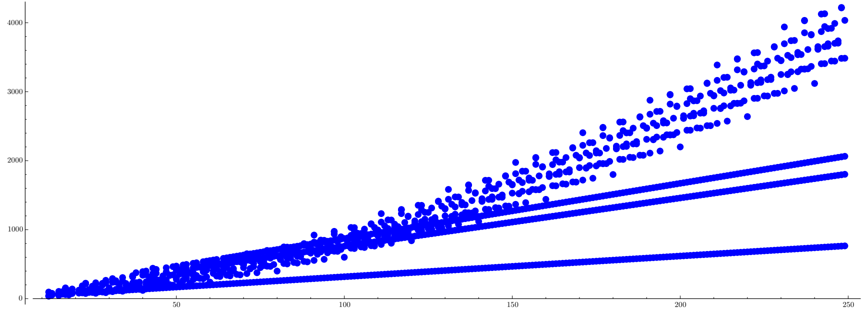

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Underlying cause: minimal presentations!

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Let $S = \langle r_1, \ldots, r_k \rangle$.

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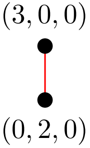

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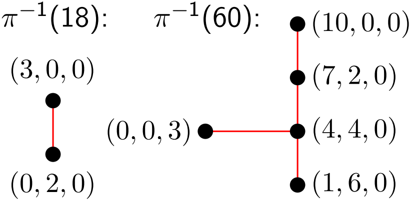

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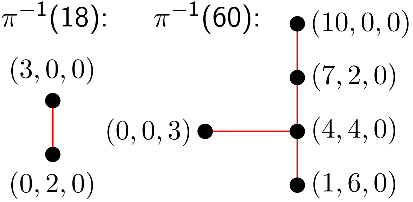

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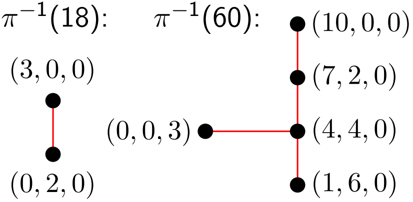

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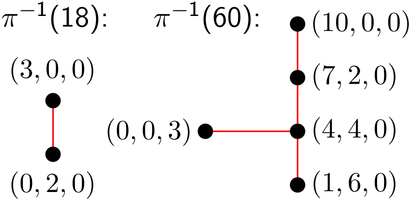

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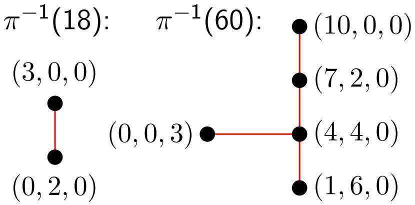

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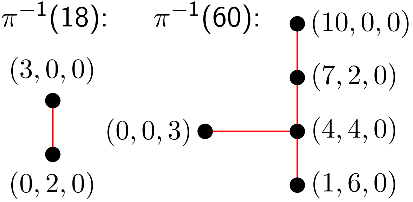

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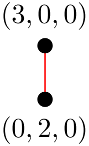

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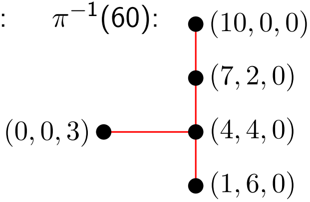

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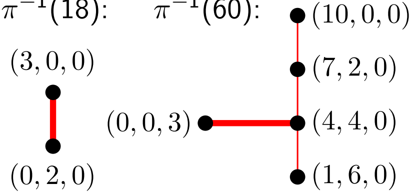

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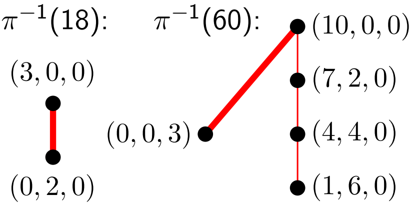

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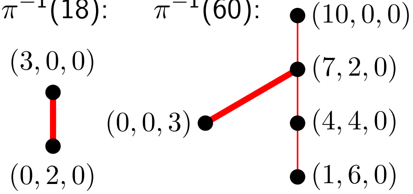

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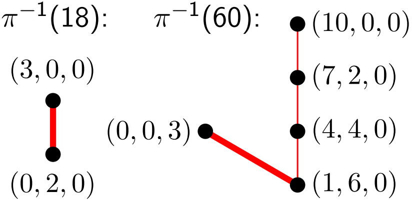


All minimal presentations:

$\{((3,0,0),(0,2,0)), ((10,7,0),(0,0,3))\}$
$\{((3,0,0),(0,2,0)), ((7,2,0),(0,0,3))\}$
$\{((3,0,0),(0,2,0)), ((4,4,0),(0,0,3))\}$
$\{((3,0,0),(0,2,0)), ((1,6,0),(0,0,3))\}$

Let $S = \langle r_1, \ldots, r_k \rangle$.

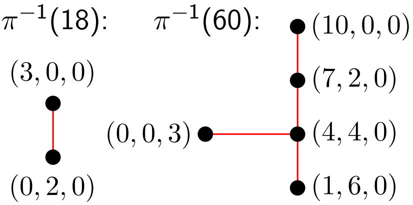
$$n = a_1 r_1 + \cdots + a_k r_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$$

$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \cdots + a_k r_k \end{aligned}$$

Definition

A *minimal presentation* ρ of S is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$: $\rho = \{((3,0,0),(0,2,0)),((4,4,0),(0,0,3))\} \subset \ker \pi$



All minimal presentations:

$\{((3,0,0),(0,2,0)),((10,7,0),(0,0,3))\}$
$\{((3,0,0),(0,2,0)),((7,2,0),(0,0,3))\}$
$\{((3,0,0),(0,2,0)),((4,4,0),(0,0,3))\}$
$\{((3,0,0),(0,2,0)),((1,6,0),(0,0,3))\}$

Kernel congruences and minimal presentations

Let $S = \langle r_1, \ldots, r_k \rangle$.

$$n = a_1 r_1 + \cdots + a_k r_k \quad \leftrightsquigarrow \quad \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{N}^k$$

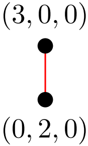
$$\begin{aligned} \pi : \mathbb{N}^k &\longrightarrow \langle r_1, \ldots, r_k \rangle \\ \mathbf{a} &\longmapsto a_1 r_1 + \cdots + a_k r_k \end{aligned}$$

Definition

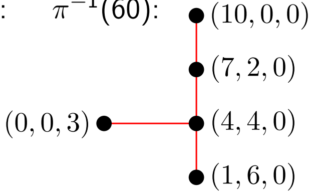
A *minimal presentation* ρ of S is a minimal subset $\rho \subset \ker \pi$ whose reflexive, symmetric, transitive, and translation closure equals $\ker \pi$.

$S = \langle 6, 9, 20 \rangle$: $\rho = \{((3,0,0),(0,2,0)),((4,4,0),(0,0,3))\} \subset \ker \pi$

$\pi^{-1}(18)$:



$\pi^{-1}(60)$:



All minimal presentations:

$\{((3,0,0),(0,2,0)),((10,7,0),(0,0,3))\}$
$\{((3,0,0),(0,2,0)),((7,2,0),(0,0,3))\}$
$\{((3,0,0),(0,2,0)),((4,4,0),(0,0,3))\}$
$\{((3,0,0),(0,2,0)),((1,6,0),(0,0,3))\}$

$$\beta_0(I_S) = \{18, 60\}$$

The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

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where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

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where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

Sneak peek for $M_n = \langle n, n + 6, n + 9, n + 20 \rangle$ and $n \gg 0$:

The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

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Sneak peek for $M_n = \langle n, n + 6, n + 9, n + 20 \rangle$ and $n \gg 0$:

M_{450}:

$$\left\{ \begin{array}{l} ((\ \ 0,0,8,0),(3,2,0,\ \ 3)),((\ \ 0,1,6,0),(4,0,0,\ \ 3)),((\ \ 0,3,0,0),(1,0,2,\ \ 0)), \\ ((20,5,0,0),(0,0,0,24)),((25,1,0,0),(0,0,4,21)),((26,0,0,0),(0,2,2,21)) \end{array} \right\}$$

The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

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Sneak peek for $M_n = \langle n, n+6, n+9, n+20 \rangle$ and $n \gg 0$:

M_{450}:

$$\left\{ \begin{array}{l} ((\ \ 0,0,8,0),(3,2,0,\ \ 3)),((\ \ 0,1,6,0),(4,0,0,\ \ 3)),((\ \ 0,3,0,0),(1,0,2,\ \ 0)), \\ ((20,5,0,0),(0,0,0,24)),((25,1,0,0),(0,0,4,21)),((26,0,0,0),(0,2,2,21)) \end{array} \right\}$$

M_{470}:

$$\left\{ \begin{array}{l} ((\ \ 0,0,8,0),(3,2,0,\ \ 3)),((\ \ 0,1,6,0),(4,0,0,\ \ 3)),((\ \ 0,3,0,0),(1,0,2,\ \ 0)), \\ ((21,5,0,0),(0,0,0,25)),((26,1,0,0),(0,0,4,22)),((27,0,0,0),(0,2,2,22)) \end{array} \right\}$$

The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

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where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

Sneak peek for $M_n = \langle n, n+6, n+9, n+20 \rangle$ and $n \gg 0$:

M_{450}:

$$\left\{ \begin{array}{l} ((\ \ 0,0,8,0),(3,2,0,\ \ 3)),((\ \ 0,1,6,0),(4,0,0,\ \ 3)),((\ \ 0,3,0,0),(1,0,2,\ \ 0)), \\ ((20,5,0,0),(0,0,0,24)),((25,1,0,0),(0,0,4,21)),((26,0,0,0),(0,2,2,21)) \end{array} \right\}$$

M_{470}:

$$\left\{ \begin{array}{l} ((\ \ 0,0,8,0),(3,2,0,\ \ 3)),((\ \ 0,1,6,0),(4,0,0,\ \ 3)),((\ \ 0,3,0,0),(1,0,2,\ \ 0)), \\ ((21,5,0,0),(0,0,0,25)),((26,1,0,0),(0,0,4,22)),((27,0,0,0),(0,2,2,22)) \end{array} \right\}$$

M_{490}:

$$\left\{ \begin{array}{l} ((\ \ 0,0,8,0),(3,2,0,\ \ 3)),((\ \ 0,1,6,0),(4,0,0,\ \ 3)),((\ \ 0,3,0,0),(1,0,2,\ \ 0)), \\ ((22,5,0,0),(0,0,0,26)),((27,1,0,0),(0,0,4,23)),((28,0,0,0),(0,2,2,23)) \end{array} \right\}$$

The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

$$(\mathbf{a}, \mathbf{a}') \longmapsto \begin{cases} (\mathbf{a}, \mathbf{a}') & |\mathbf{a}| = |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_k, \mathbf{a}' + \ell \mathbf{e}_0) & |\mathbf{a}| < |\mathbf{a}'| \\ (\mathbf{a} + \ell \mathbf{e}_0, \mathbf{a}' + \ell \mathbf{e}_k) & |\mathbf{a}| > |\mathbf{a}'| \end{cases}$$

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- Φ_n is well-defined.

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where $\ell = \big||\mathbf{a}| - |\mathbf{a}'|\big|$.

- Φ_n is well-defined.

$$\begin{aligned} \pi_n(\mathbf{a}) = a_0 n + \textstyle\sum_{i=1}^k a_i(n + r_i) &= |\mathbf{a}|n + \textstyle\sum_{i=1}^k a_i r_i \\ \pi_{n+r_k}(\mathbf{a}) = \qquad\qquad\qquad &= |\mathbf{a}|n + |\mathbf{a}|r_k + \textstyle\sum_{i=1}^k a_i r_i \end{aligned}$$

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- Φ_n preserves reflexive and symmetric closure.

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- Φ_n preserves reflexive and symmetric closure.
- Φ_n preserves translation closure.

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- Φ_n preserves reflexive and symmetric closure.
- Φ_n preserves translation closure.

$$\Phi_n((\mathbf{a}, \mathbf{a}') + (\mathbf{b}, \mathbf{b})) = \Phi_n(\mathbf{a}, \mathbf{a}') + (\mathbf{b}, \mathbf{b})$$

The shifting map

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

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- Φ_n is well-defined.

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- Φ_n preserves reflexive and symmetric closure.
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$$\Phi_n((\mathbf{a}, \mathbf{a}') + (\mathbf{b}, \mathbf{b})) = \Phi_n(\mathbf{a}, \mathbf{a}') + (\mathbf{b}, \mathbf{b})$$

- Only missing link: transitivity.

Monotone chains

$$M_n = \langle n, n + r_1, \ldots, n + r_k \rangle, \qquad \pi_n : \mathbb{N}^{k+1} \longrightarrow M_n$$

Shifting map $\Phi_n : \ker \pi_n \longrightarrow \ker \pi_{n+r_k}$ given by

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where $\ell = \big| |\mathbf{a}| - |\mathbf{a}'| \big|$.

Fix $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \ker \pi_n$ with

$$|\mathbf{a}| < |\mathbf{b}| < |\mathbf{c}|.$$

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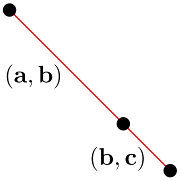

Monotone chains

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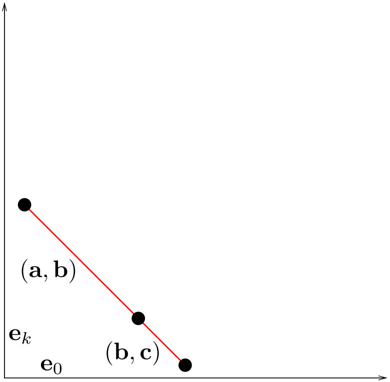

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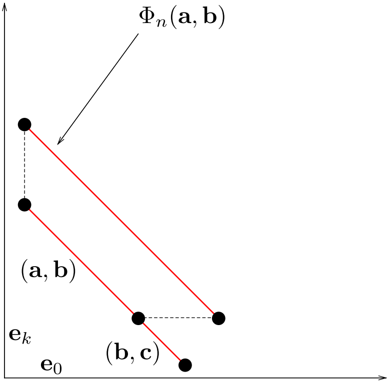

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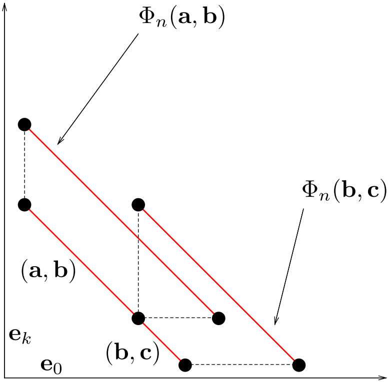

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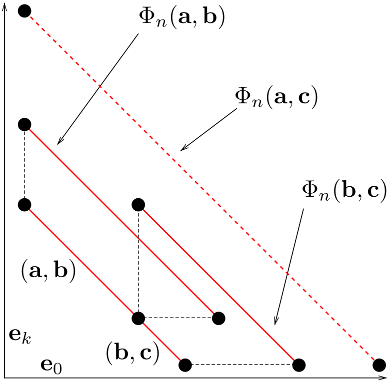

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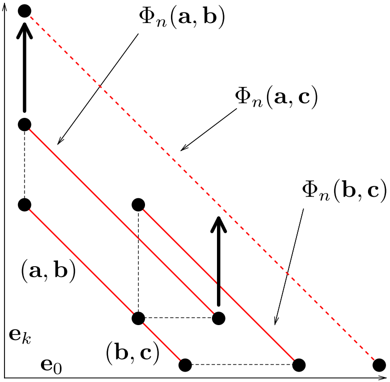

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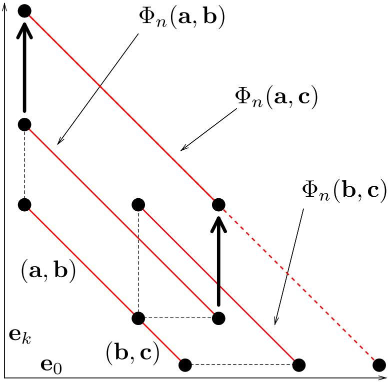

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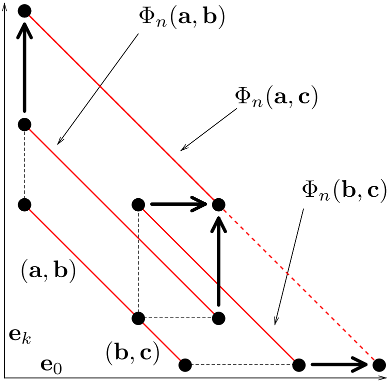

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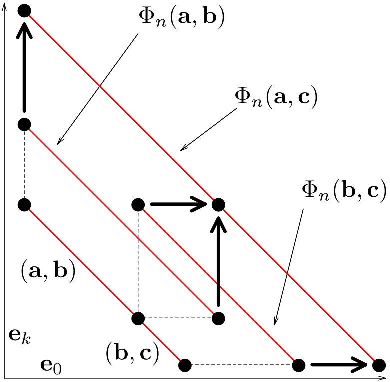

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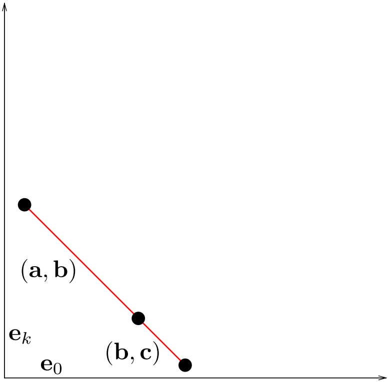

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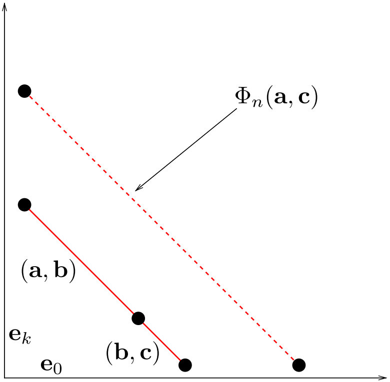

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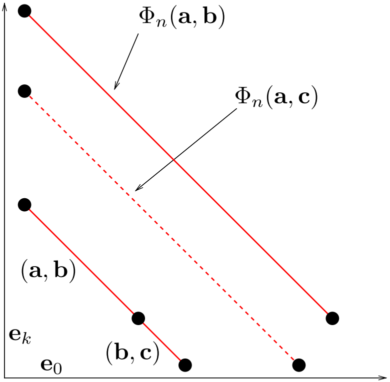

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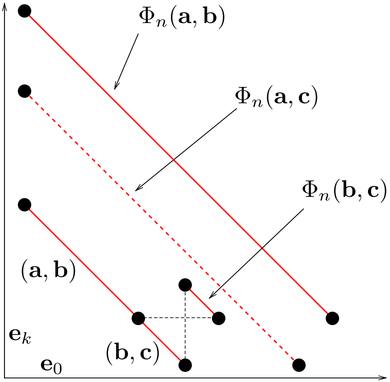

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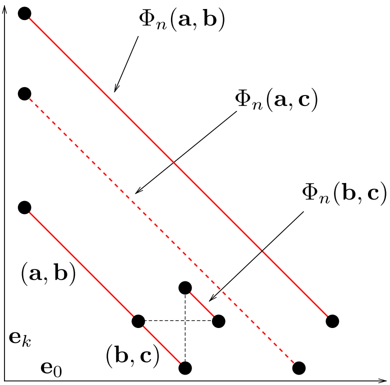
Need: *monotone* chains are sufficient for transitive closure.



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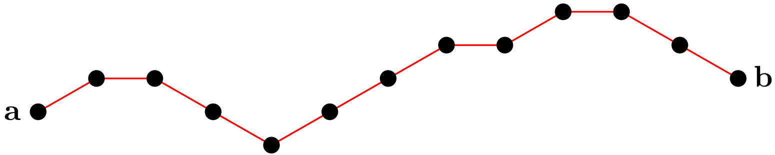

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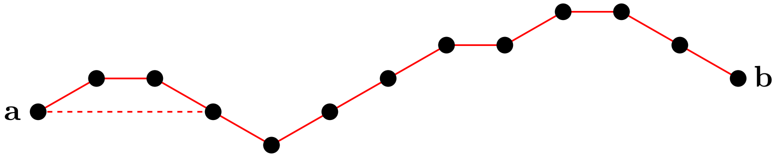

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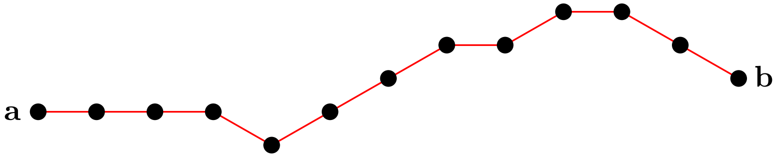

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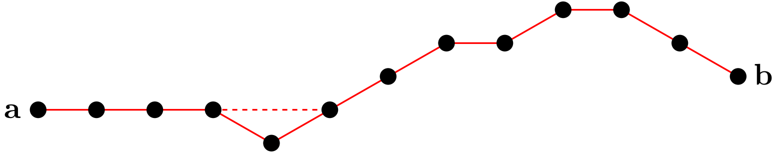

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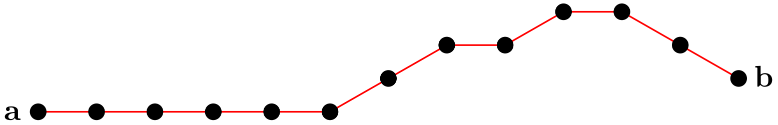

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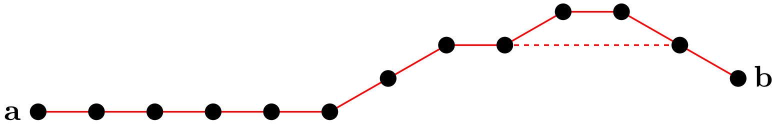

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If $n > r_k^2$ and $(\mathbf{a}, \mathbf{a}') \in \rho$ with $|\mathbf{a}| > |\mathbf{a}'|$, then $a_0 > 0$ and $a_k' > 0$.

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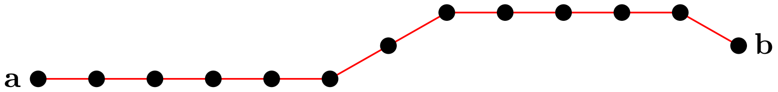

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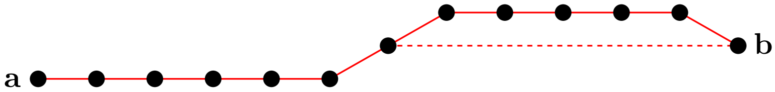

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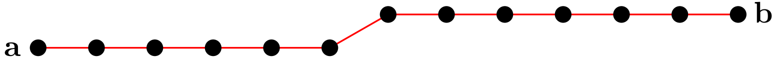

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Consequences:

- The Betti numbers $n \mapsto \beta_0(M_n)$ are eventually r_k-periodic:

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Application: computing minimal presentations

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$((21,1,0,0),(0,0,0,21)),$
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n	M_n	Min. Pres. Runtime
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200	$\langle 200, 206, 209, 220 \rangle$	40 ms
400	$\langle 400, 406, 409, 420 \rangle$	210 ms
1000	$\langle 1000, 1006, 1009, 1020 \rangle$	3 sec
3000	$\langle 3000, 3006, 3009, 3020 \rangle$	2 min
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GAP Numerical Semigroups Package, available at

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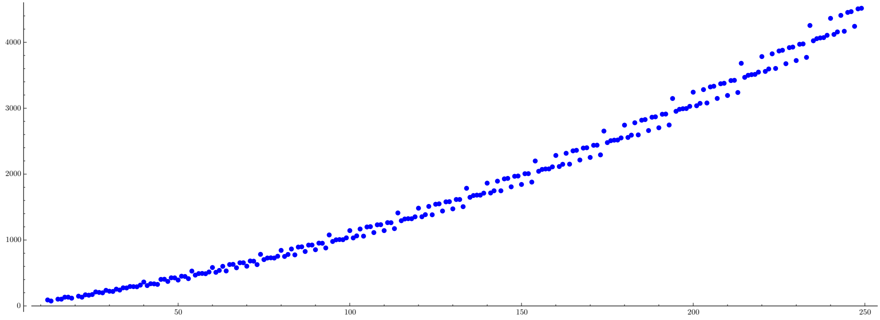

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